Tutorial 5: Inverse Kinematics

These questions are from the Practice Exercises of the Modern Robotics book. The solutions can be found on the book website. Please try your best before referring to the solutions.

Question 1: Newton-Raphson root finding

Perform three iterations of (approximate) iterative Newton-Raphson root finding on the scalar function $x_d - f(\theta)$ in *Figure 1*, starting from θ^0. (A general vector function $f(\theta)$ could represent the forward kinematics of a robot, and x_d could represent the desired configuration in coordinates. The roots of $x_d - f(\theta)$ are the joint vectors θ satisfying $x_d - f(\theta) = 0$, i.e., solutions to the inverse kinematics problem.) Draw the iterates θ^1, θ^2 and θ^3 on the θ axis and illustrate clearly how you obtain these points.

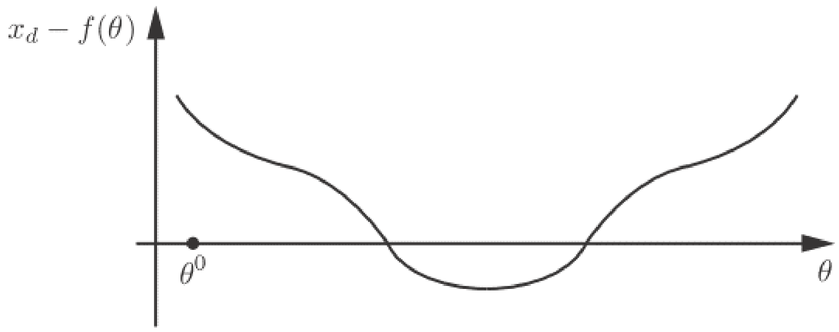


Figure 1. A scalar function $x_d - f(\theta)$ of θ.

Question 2: RRP robot arm

The spatial RRP open chain of *Figure 2* is shown in its zero position. Use analytic methods to solve the inverse kinematics when the end-effector configuration is described by

$$T = \begin{bmatrix} 0 & 1 & 0 & 2L \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -3L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

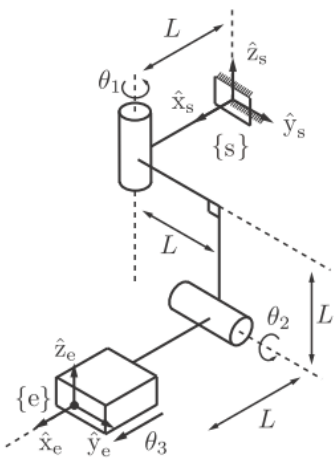


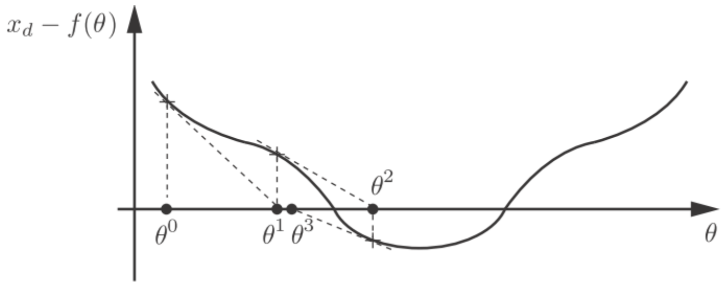
Figure 2. An RRP Robot.

Sample solution

Ref: Modern Robotics Practice Exercises

Question 1: Newton-Raphson root finding

This question for understanding the steps involved in updating the guess of θ using the gradient at each value of θ^i.

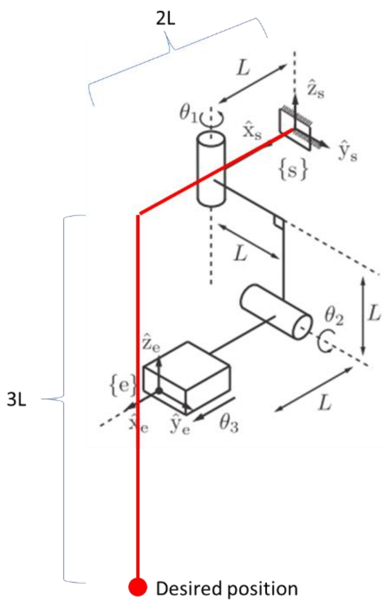


Question 2: RRP robot arm

From the given pose, we see that the desired position (translation) is $\begin{bmatrix} 2L \\ 0 \\ -3L \end{bmatrix}$, i.e. a translation of $2L$ in $\hat{x}_s$ direction, and $3L$ in $-\hat{z}_s$ direction.

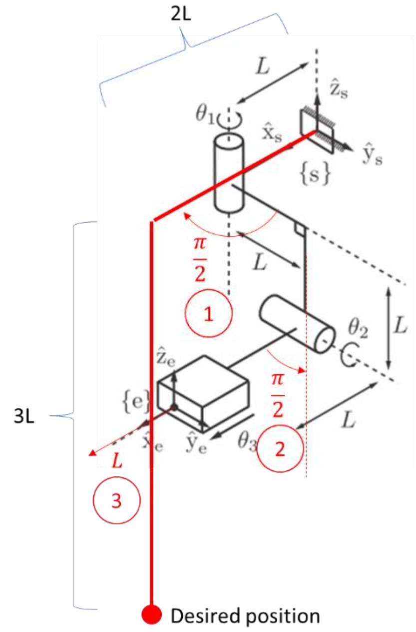
$$T = \begin{bmatrix} 0 & 1 & 0 & 2L \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -3L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We also note the robot arm has 3 dof, so we can only control the position (3 dof), and the orientation will be dependent on the position. Below diagram shows the desired position of the end-effector.



To find θ_1, θ_2 and θ_3 (solve IK problem) analytically we may solve by geometry or algebra, or a combination of both.

By geometry, we can visually find the solution by taking the motions of ① $\theta_1 = -\frac{\pi}{2}$ (note the positive direction as per RHR), then ② $\theta_2 = \frac{\pi}{2}$, and finally ③ $\theta_3 = L$ (note this is a prismatic joint).



$$\theta = \left(-\frac{\pi}{2}, \frac{\pi}{2}, L\right)$$

Alternatively, we can solve linear equations by formulating the FK equations and solve for the θ. In this case we will determine the FK for the end-effector pose wrt {s}, $T_{se}(\theta)$ and equate it to the desired pose T.

Recall we can form FK equations by (1) geometry, (2) homogeneous transformation matrix, (3) power of exponentials. In this sample solution, we are using (2). It is certainly a better idea to use computer to solve the IK problem. However for the purpose of learning the concepts, you should know how to solve IK manually on paper, for simple robotic arms.

$$T_{se} = trans(\hat{x}_s, L)rot(\hat{z}_s, \theta_1)trans(\hat{y}_s, L)trans(-\hat{z}_s, L)rot(\hat{y}_s, \theta_2)trans(\hat{x}_s, L + \theta_3)$$

$$① \; trans(\hat{x}_s, L) = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times rot(\hat{z}_s, \theta_1) = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & -s_1 & 0 & L \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$② \; trans(\hat{y}_s, L) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times trans(-\hat{z}_s, L) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

③ $rot(\hat{y}_s, \theta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times trans(\hat{x}_s, L+\theta_3) = \begin{bmatrix} 1 & 0 & 0 & L+\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$

$$\begin{bmatrix} c_2 & 0 & s_2 & c_2(L+\theta_3) \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{se} = ① \times ② \times ③ = \begin{bmatrix} c_1 & -s_1 & 0 & L \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 & c_2(L+\theta_3) \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & -s_1 & 0 & -s_1 L + L \\ s_1 & c_1 & 0 & c_1 L \\ 0 & 0 & 1 & -L \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & s_2 & c_2(L+\theta_3) \\ 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 c_2(L+\theta_3) - s_1 L + L \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 c_2(L+\theta_3) + c_1 L \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3) - L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equating the FK equation of T_{se} and the desired T_{se}, we solve $T_{se} = T$ for θ_1, θ_2 and θ_3.

$$T_{se} = T$$

$$\begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 c_2(L+\theta_3) - s_1 L + L \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 c_2(L+\theta_3) + c_1 L \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3) - L \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2L \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -3L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can choose any element pair, e.g.

$$\begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 c_2(L+\theta_3) - s_1 L + L \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 c_2(L+\theta_3) + c_1 L \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3) - L \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 & 0 & 2L \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -3L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-\sin\theta_1 = 1, \text{ i.e. } \sin\theta_1 = -1 \text{ gives } \theta_1 = -\frac{\pi}{2}$$

We can confirm with other elements with θ_1, e.g. $\cos\theta_1 = 0$ if $\theta_1 = -\frac{\pi}{2}$.

Likewise, we solve for θ_2,

$$\begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 c_2(L+\theta_3) - s_1 L + L \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 c_2(L+\theta_3) + c_1 L \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3) - L \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2L \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -3L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-\sin\theta_2 = -1, \text{ i.e. } \sin\theta_2 = 1 \text{ gives } \theta_2 = \frac{\pi}{2}$$

Confirm: $\cos\theta_2 = 0$ if $\theta_2 = -\frac{\pi}{2}$.

Finally, we find equations to solve for θ_3 knowing θ_1 and θ_2,

$$\begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1c_2(L+\theta_3)-s_1L+L \\ s_1c_2 & c_1 & s_1s_2 & s_1c_2(L+\theta_3)+c_1L \\ -s_2 & 0 & c_2 & -s_2(L+\theta_3)-L \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2L \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -3L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-s_2(L+\theta_3)-L=-3L$$

$$-1(L+\theta_3)-L=-3L$$

$$\theta_3 = L$$