Tutorial 4: Velocity Kinematics

These questions are from the Practice Exercises of the Modern Robotics book. The solutions can be found on the book website. Please try your best before referring to the solutions.

Question 1: KUKA LBR iiwa 7R robot arm

Figure 1 shows the KUKA LBR iiwa 7R robot arm. The figure defines an $\{s\}$ frame at the base with the $\hat{y}_s$-axis pointing out of the page and a $\{b\}$ frame aligned with $\{s\}$ at the end-effector. The robot is at its home configuration. The screw axes for the seven joints are illustrated (positive rotation about these axes is by the right-hand rule). The axes for joints 2, 4, and 6 are aligned, and the axes for joints 1, 3, 5, and 7 are identical at the home configuration. The dimensions are $L_1 = 0.34\ m$, $L_2 = 0.4\ m$, $L_3 = 0.4\ m$, and $L_4 = 0.15\ m$.

a. What is the space Jacobian when the robot is at its home configuration?
b. Assume the angles of the joints are ${}^{i\pi}/_{16}$ for joints $i = 1 \cdots 7$. What is the space Jacobian?

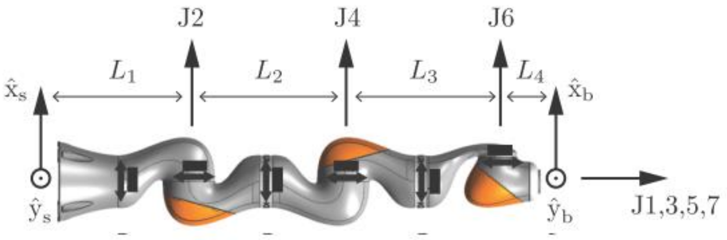


Figure 1. The KUKA LBR iiwa 7-dof robot.

Question 2: RPR robot arm

Figure 2 shows an RPR robot that is confined to the plane of the page. An end-effector frame $\{b\}$ is illustrated, where the $\hat{x}_b$-axis is out of the page. The directions of positive motion of the three joints are indicated by arrows. The axes of the two revolute joints are out of the page, and the prismatic joint moves in the plane of the page. Joint 1 is at $q_1 = (0, -5, 7)$ in $\{b\}$ and joint 3 is at $q_3 = (0, -1, -3)$ in $\{b\}$. Write the body Jacobian $J_b(\theta)$ for the configuration shown. All entries of your $J_b(\theta)$ matrix should be numerical (no symbols or math).

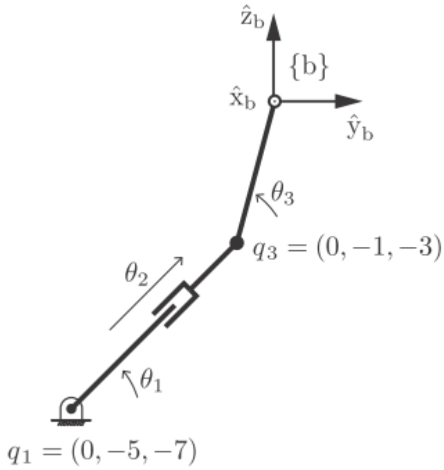


Figure 2. An RPR Robot.

Sample solution

Questions are from Modern Robotics Practice Exercises.

Question 1: KUKA LBR iiwa 7R robot arm

The sample solutions are purposely elaborated to show the calculations involved. They have been manually worked out and have not been verified. The sample answers may have errors; they are meant to show the steps involved.

In real scenarios, these calculations are done on a computer. However, for the purpose of the assessments in this module, you are expected to be able to perform the calculations manually for simple robotic arms.

First note there are 7 joints, so n=7, the Jacobian matrix will have 7 columns. We take note of the whereabout of J1 to J7. Since we are interested in space Jacobian, we begin with the joint closest to the {s} frame, i.e. J1 and progress further away from {s}, i.e. J2, J3, .. J7. Taking note that as we move to further joints, the motion of preceding joints will affect the current joint being considered. E.g. for J4, the ω_4 and v_4 will be affected by the values of θ_1, θ_2 and θ_3, i.e. motion of preceding joints.

Two approaches:

1. We may derive a general expression for each column of the Jacobian matrix in terms of $\boldsymbol{\theta}$. The expression can be used to determine the Jacobian matrix given any values of $\boldsymbol{\theta}$. For (a), we substitute $\boldsymbol{\theta} = [0]$. For (b), we substitute $\boldsymbol{\theta} = \left[\theta_i = {}^{i\pi}/_{16}\ for\ i = 1 \ldots 7\right]$.
2. Alternatively, we the computations can use the values of $\boldsymbol{\theta}$ to determine each ω_i and v_i from the beginning without trying to find the general expression. We can use rotation matrices to account for the rotation induced by earlier joints.

The sample solution below uses the second approach.

<u>Question 1(a): Exercise 5.1(a)</u>

Note this question determines the Jacobian at a specific configuration, the zero/home configuration at $\boldsymbol{\theta} = [0]$. Therefore, the final answer of the Jacobian is not in the function of $\boldsymbol{\theta}$.

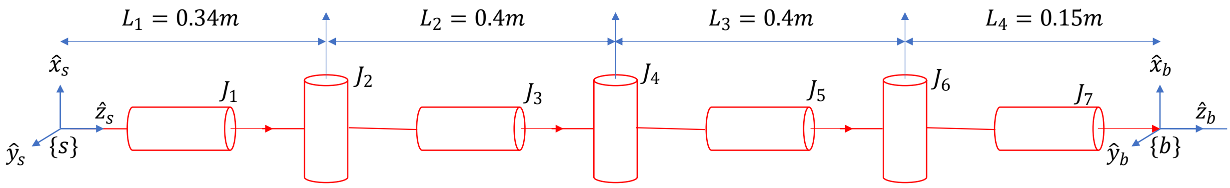


Figure 3. Zero/home configuration.

At home configuration,

$$J_s^0 = [J_{s1}^0 \quad J_{s2}^0 \quad J_{s3}^0 \quad J_{s4}^0 \quad J_{s5}^0 \quad J_{s6}^0 \quad J_{s7}^0]$$

We will determine each Jacobian column.

$$J_{s1}^0 = \begin{bmatrix} \omega_1^0 \\ v_1^0 \end{bmatrix} \qquad \omega_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ in the direction of } J_1\text{, i.e. } \hat{z}_s.$$

$$v_1^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ no linear motion on \{s\}.}$$

$$J_{s2}^0 = \begin{bmatrix} \omega_2^0 \\ v_2^0 \end{bmatrix} \qquad \omega_2^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ in the direction of } J_2\text{, i.e. } \hat{x}_s.$$

$$v_2^0 = \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.34 \\ 0 \end{bmatrix}$$

when rotate around J_2, {s} experiences a linear motion in the direction of $\hat{y}_s$ at a distance of L_1.

$$J_{s3}^0 = \begin{bmatrix} \omega_3^0 \\ v_3^0 \end{bmatrix} \qquad \omega_3^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ in the direction of } J_3\text{, i.e. } \hat{z}_s.$$

$$v_3^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rotating around J_3 does not cause linear motion on {s}.

$$J_{s4}^0 = \begin{bmatrix} \omega_4^0 \\ v_4^0 \end{bmatrix} \qquad \omega_4^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ in the direction of } J_4\text{, i.e. } \hat{x}_s.$$

$$v_4^0 = \begin{bmatrix} 0 \\ L_1 + L_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.74 \\ 0 \end{bmatrix}$$

rotating around J_4 causes {s} to move (linear) in the direction of $\hat{y}_s$ according to RHR (right hand rule), at a distance of $L_1 + L_2$.

$$J_{s5}^0 = \begin{bmatrix} \omega_5^0 \\ v_5^0 \end{bmatrix} \qquad \omega_5^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_5^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

similar to J_3.

$$J_{s6}^0 = \begin{bmatrix} \omega_6^0 \\ v_6^0 \end{bmatrix} \qquad \omega_6^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_6^0 = \begin{bmatrix} 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.14 \\ 0 \end{bmatrix}$$

similar to J_4.

$$J_{s7}^0 = \begin{bmatrix} \omega_7^0 \\ v_7^0 \end{bmatrix} \qquad \omega_7^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_7^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

similar to J_1, J_3, J_5.

<u>Question 1(b): Exercise 5.1(b)</u>

One way to find ω_i and v_i at an arbitrary angle $\boldsymbol{\theta}$ is to apply the transformation caused by $\theta_{i-1} \dots \theta_1$, i.e. preceding joints.

Given $\theta_i = {}^{i\pi}/_{16}$, we obtain

$$\begin{aligned} \theta &= (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7) \\ &= \left(\frac{\pi}{16}, \frac{2\pi}{16}, \frac{3\pi}{16}, \frac{4\pi}{16}, \frac{5\pi}{16}, \frac{6\pi}{16}, \frac{7\pi}{16}\right) \\ &= \left(\frac{\pi}{16}, \frac{\pi}{8}, \frac{3\pi}{16}, \frac{\pi}{4}, \frac{5\pi}{16}, \frac{3\pi}{8}, \frac{7\pi}{16}\right) \end{aligned}$$

We want to determine

$$J_s(\theta) = \begin{bmatrix} J_{s1} & J_{s2} & J_{s3} & J_{s4} & J_{s5} & J_{s6} & J_{s7} \end{bmatrix}$$

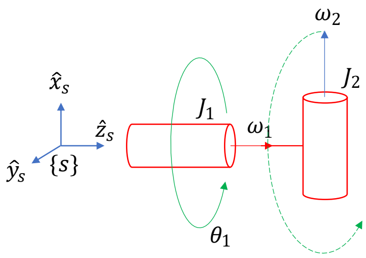
We will determine each Jacobian column.

$J_{s1} = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix}$ The axis of J_1, i.e. ω_1 is not affected by any θ_i, it remains at

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Likewise, rotating around it does not cause any linear motion on {s},

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$



Notice the direction of J_2, i.e. ω_2 is affected by (dependent on) θ_1 (preceding joint).

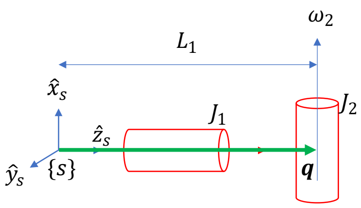
We can determine ω_2 by applying the rotation of θ_1 on ${\omega_2}^0$.

$$
\begin{aligned}
\omega_2 &= rot(\hat{z}_s, \theta_1){\omega_2}^0 \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos\frac{\pi}{16} \\ \sin\frac{\pi}{16} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.999\ldots \\ 0.00343 \\ 0 \end{bmatrix} \cong \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; given\ \theta_1 = \frac{\pi}{16}
\end{aligned}
$$

Likewise, we apply the rotation due to θ_1 on ${v_2}^0$.

$$
\begin{aligned}
v_2 &= rot(\hat{z}_s, \theta_1){v_2}^0 \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -L_1\sin\theta_1 \\ L_1\cos\theta_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -0.34\sin\theta_1 \\ 0.34\cos\theta_1 \\ 0 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0.34 \\ 0 \end{bmatrix} \; given\ \theta_2 = \frac{\pi}{8}\ and\ L_1 = 0.34
\end{aligned}
$$

Note, we can alternatively determine v_2 using cross product,

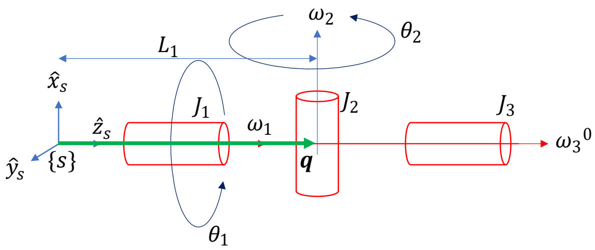


$q = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$

(expressed in {s})

$$
\begin{aligned}
v_2 &= -\omega_2 \times q \\
&= -\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} \\
&= \begin{bmatrix} -L_1\sin\theta_1 \\ L_1\cos\theta_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
J_{s2} = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \cong \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0.34 \\ 0 \end{bmatrix}
$$



We considering ω_3 and v_3, we note they are transformed from $\omega_3{}^0$ and $v_3{}^0$ by θ_1 and θ_2.

We can apply the transformation on $\omega_3{}^0$ and $v_3{}^0$ to determine ω_3 and v_3.

$$\begin{aligned}
\omega_3 &= rot(\hat{z}_s, \theta_1) rot(\hat{x}_s, \theta_2) \omega_3{}^0 \\
&= R_1 rot(\hat{x}_s, \theta_2) \omega_3{}^0 \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2 \\ \sin\theta_1 & \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} s_1 s_2 \\ -c_1 s_2 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} 0.0000234\ldots \\ 0.00685\ldots \\ 0.999\ldots \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0.01 \\ 1 \end{bmatrix} \; given\ \theta_1 = \frac{\pi}{16}, \theta_2 = \frac{2\pi}{16} = \frac{\pi}{8}
\end{aligned}$$

$$\begin{aligned}
v_3 &= rot(\hat{z}_s, \theta_1) rot(\hat{x}_s, \theta_2) v_3{}^0 \\
&= R_2 v_3{}^0 \\
&= \begin{bmatrix} c_1 & -s_1 c_2 & s_1 s_2 \\ s_1 & c_1 c_2 & -c_1 s_2 \\ 0 & s_2 & c_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

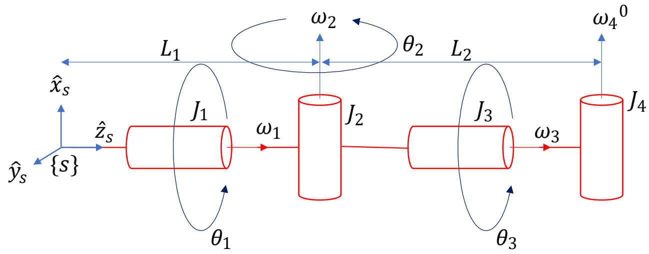
Alternatively,

$$\begin{aligned}
v_3 &= -\omega_3 \times q \\
&= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Note q is defined from {s} to the intersection of ω_2 and ω_3. Basically q can be from {s} to any point on ω_3. Since ω_3 intersects at ω_2, this point is convenient.

$$J_{s3} = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For J_{s4},

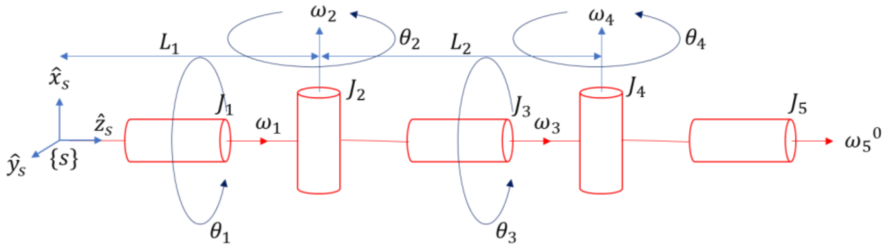


$$\begin{aligned}\omega_4 &= rot(\hat{z}_s,\theta_1)rot(\hat{x}_s,\theta_2)rot(\hat{z}_s,\theta_3)\omega_4{}^0\\ &= R_2 rot(\hat{z}_s,\theta_3)\omega_4{}^0\\ &= \begin{bmatrix} c_1 & -s_1c_2 & s_1s_2\\ s_1 & c_1c_2 & -c_1s_2\\ 0 & s_2 & c_2\end{bmatrix}\begin{bmatrix} c_3 & -s_3 & 0\\ s_3 & c_3 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\\ &= \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3 - s_1c_2c_3 & s_1s_2\\ s_1c_3 + c_1c_2s_3 & -s_1s_3 + c_1c_2c_3 & -c_1s_2\\ s_2s_3 & s_2c_3 & c_2\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = R_3\begin{bmatrix}1\\0\\0\end{bmatrix}\\ &= \begin{bmatrix} c_1c_3 - s_1c_2s_3\\ s_1c_3 + c_1c_2s_3\\ s_2s_3\end{bmatrix}\\ &= \begin{bmatrix} 0.999\ldots\\ 0.0137\ldots\\ 0.00007\ldots\end{bmatrix} \cong \begin{bmatrix}1\\0.014\\0\end{bmatrix}\ given\ \theta_1 = \frac{\pi}{16}, \theta_2 = \frac{2\pi}{16} = \frac{\pi}{8}, \theta_3 = \frac{3\pi}{16}\end{aligned}$$

$$\begin{aligned}v_4 &= rot(\hat{z}_s,\theta_1)rot(\hat{x}_s,\theta_2)rot(\hat{z}_s,\theta_3)v_4{}^0\\ &= R_3 v_4{}^0\\ &= \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3 - s_1c_2c_3 & s_1s_2\\ s_1c_3 + c_1c_2s_3 & -s_1s_3 + c_1c_2c_3 & -c_1s_2\\ s_2s_3 & s_2c_3 & c_2\end{bmatrix}\begin{bmatrix}0\\L_1+L_2\\0\end{bmatrix}\\ &= \begin{bmatrix} -c_1s_3(L_1+L_2) - s_1c_2s_3(L_1+L_2)\\ -s_1s_3(L_1+L_2) + c_1c_2c_3(L_1+L_2)\\ s_2c_3(L_1+L_2)\end{bmatrix}\\ &= \begin{bmatrix} -1 * 0.01028\ldots(L_1+L_2) - 0\\ 0 + 1 * 0.999\ldots(L_1+L_2)\\ 0\end{bmatrix}\ given\ \theta_1 = \frac{\pi}{16}, \theta_2 = \frac{2\pi}{16} = \frac{\pi}{8}, \theta_3 = \frac{3\pi}{16}\\ &= \begin{bmatrix} -0.0076\ldots\\ 0.739\ldots\\ 0\end{bmatrix} \cong \begin{bmatrix}0\\0.74\\0\end{bmatrix}\ given\ \ L_1 = 0.34, L_2 = 0.4, L_1 + L_2 = 0.74m\end{aligned}$$

$$J_{s4} = \begin{bmatrix}\omega_4\\ v_4\end{bmatrix} \cong \begin{bmatrix}1\\0.014\\0\\0\\0.74\\0\end{bmatrix}$$

For J_{s5},

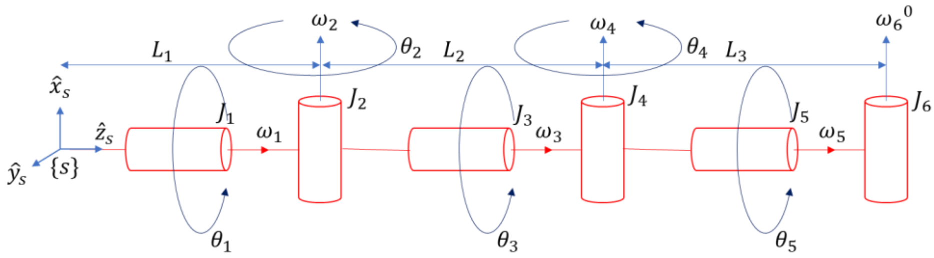


$$\begin{aligned}
\omega_5 &= rot(\hat{z}_s,\theta_1)rot(\hat{x}_s,\theta_2)rot(\hat{z}_s,\theta_3)rot(\hat{x}_s,\theta_4)\omega_5{}^0 \\
&= R_3 rot(\hat{x}_s,\theta_4)\omega_5{}^0 \\
&= \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3 - s_1c_2c_3 & s_1s_2 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3 + c_1c_2c_3 & -c_1s_2 \\ s_2s_3 & s_2c_3 & c_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_4 & -s_4 \\ 0 & s_4 & c_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3c_4 - s_1c_2c_3c_4 + s_1s_2s_4 & c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3c_4 + c_1c_2c_3c_4 - c_1s_2s_4 & s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ s_2s_3 & s_2c_3c_4 + c_2s_4 & -s_2c_3s_4 + c_2c_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ -s_2c_3s_4 + c_2c_4 \end{bmatrix} \, note\ most\ of\ the \cos and \sin are\ either\ 0\ or\ 1 \\
&\cong \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \, given\ \theta_1 = \frac{\pi}{16}, \theta_2 = \frac{2\pi}{16} = \frac{\pi}{8}, \theta_3 = \frac{3\pi}{16}, \theta_4 = \frac{4\pi}{16} = \frac{\pi}{4}
\end{aligned}$$

$$v_5 = R_4 v_5{}^0 = R_4 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s5} = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For J_{s6},

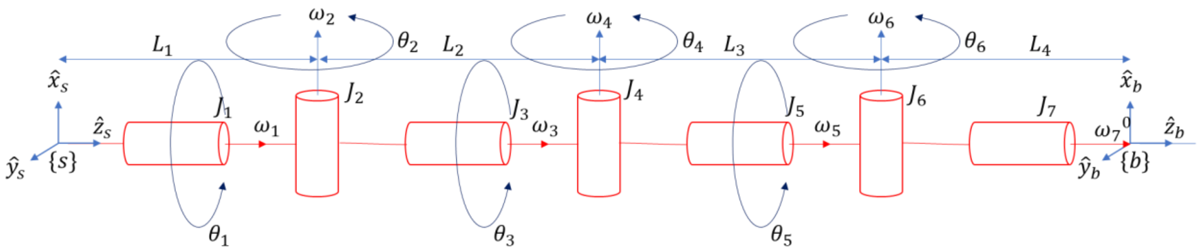


$$\begin{aligned}
\omega_6 &= R_4 rot(\hat{z}_s, \theta_5)\omega_6{}^0 \\
&= R_4 \begin{bmatrix} c_5 & -s_5 & 0 \\ s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= R_4 \begin{bmatrix} c_5 \\ s_5 \\ 0 \end{bmatrix} = R_4 \begin{bmatrix} 0.999\ldots \\ 0.017 \\ 0 \end{bmatrix} \; given\ \theta_5 = \frac{5\pi}{16} \\
&\cong R_4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&\cong \begin{bmatrix} c_1c_3 - s_1c_2s_3 \\ s_1c_3 + c_1c_2s_3 \\ s_2s_3 \end{bmatrix} \; column\ 1\ of\ R_4 \\
&\cong \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; given\ \theta_1 = \frac{\pi}{16}, \theta_2 = \frac{2\pi}{16} = \frac{\pi}{8}, \theta_3 = \frac{3\pi}{16}
\end{aligned}$$

$$\begin{aligned}
v_6 &= R_4 rot(\hat{z}_s, \theta_5)v_6{}^0 \\
&= R_4 \begin{bmatrix} c_5 & -s_5 & 0 \\ s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix} \\
&= R_4 \begin{bmatrix} -s_5(L_1 + L_2 + L_3) \\ c_5(L_1 + L_2 + L_3) \\ 0 \end{bmatrix} \\
&\cong R_4 \begin{bmatrix} 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix} \; given\ \theta_5 = \frac{5\pi}{16}, s_5 \cong 0, c_5 \cong 1 \\
&\cong (L_1 + L_2 + L_3) \begin{bmatrix} -c_1s_3c_4 - s_1c_2c_3c_4 + s_1s_2s_4 \\ -s_1s_3c_4 + c_1c_2c_3c_4 - c_1s_2s_4 \\ s_2c_3c_4 + c_2s_4 \end{bmatrix} \; column\ 2\ of\ R_4 \\
&\cong (L_1 + L_2 + L_3) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; given\ \theta_1 = \frac{\pi}{16}, \theta_2 = \frac{2\pi}{16} = \frac{\pi}{8}, \theta_3 = \frac{3\pi}{16}, \theta_4 = \frac{4\pi}{16} = \frac{\pi}{4} \\
&\cong \begin{bmatrix} 0 \\ 1.14 \\ 0 \end{bmatrix} \; given\ L_1 = 0.34, L_2 = 0.4, L_3 = 0.4m
\end{aligned}$$

$$J_{s6} = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} \cong \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1.14 \\ 0 \end{bmatrix}$$

For J_{s7},



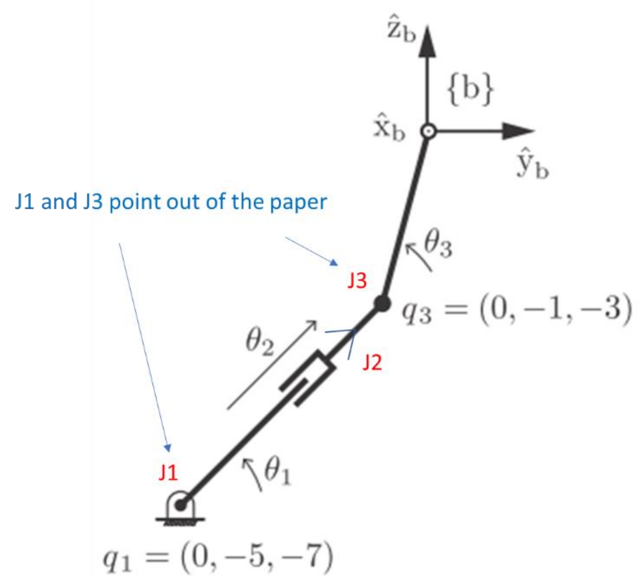
$$\begin{aligned}
\omega_7 &= R_4 rot(\hat{z}_s, \theta_5) rot(\hat{x}_s, \theta_6) \omega_7{}^0 \\
&= R_4 \begin{bmatrix} c_5 & -s_5 & 0 \\ s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_6 & -s_6 \\ 0 & s_6 & c_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= R_4 \begin{bmatrix} c_5 & -s_5 c_6 & s_5 s_6 \\ s_5 & c_5 c_6 & -c_5 s_6 \\ 0 & s_6 & c_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= R_4 \begin{bmatrix} s_5 s_6 \\ -c_5 s_6 \\ c_6 \end{bmatrix} \\
&\cong R_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \; given \; \theta_6 = \frac{6\pi}{16} = \frac{3\pi}{8}, s_6 \cong 0, c_6 \cong 1 \\
&\cong \begin{bmatrix} c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} \\
&\cong \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \; given \; \theta_1 = \frac{\pi}{16}, \theta_2 = \frac{2\pi}{16} = \frac{\pi}{8}, \theta_3 = \frac{3\pi}{16}, \theta_4 = \frac{4\pi}{16} = \frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
v_7 &= R_4 rot(\hat{z}_s, \theta_5) rot(\hat{x}_s, \theta_6) v_7{}^0 \\
&= R_4 \begin{bmatrix} c_5 & -s_5 & 0 \\ s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_6 & -s_6 \\ 0 & s_6 & c_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

$$J_{s7} = \begin{bmatrix} \omega_7 \\ v_7 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Question 2: RPR robot arm

(Exercise 5.3)



There are three joints, n = 3, the Jacobian matrix has 3 columns.

$$J_b(\theta) = [J_{b1} \quad J_{b2} \quad J_{b3}]$$

Since we are looking for body Jacobian, we start from the joint closest to the {b} frame and progress to joints further from {b}, i.e. Joint 3, then 2, then 1. We take note the Jacobian column of a joint will be affected by the motion of "preceding" joints. In the case of body Jacobian, "preceding" means closer to the {b} frame.

From the diagram, we can see that ω_1 (J1) and ω_3 (J3) are in the direction of $\hat{x}_b$ (note the reference is {b} frame) taking note of the direction of θ_1 and θ_3 in RHR. We also note J2 is a prismatic joint at a direction along the first link.

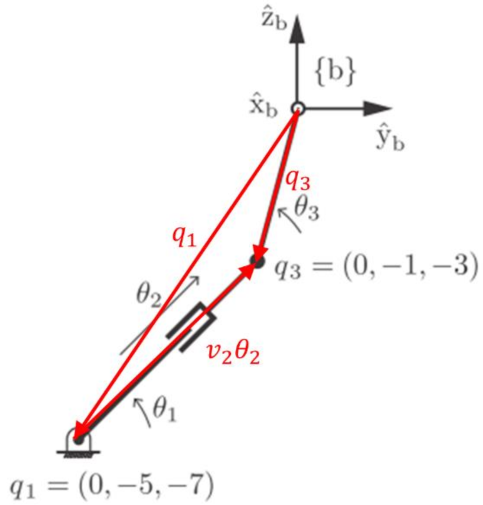
We now find $J_{b3} = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix}$. We already note $\omega_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ since J3 is in the direction of $\hat{x}_b$. We have been given the vector q3 connecting J3 and {b}, we find

$$v_3 = -\omega_3 \times q_3 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Next we find $J_{b2} = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix}$.

J2 is a prismatic joint, so $\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. v_2 is the direction (with reference to {b}) of the prismatic joint. We can see that it changes (with reference to {b}) when θ_3 change. It should be a function of θ_3. However for the question, θ_3 is not given, and the question ask for the answer in numerical, i.e. not in a function of θ_3, for e.g. Instead we know q_1 and q_3. We can find v_2 from vector addition of q_1 and q_3.

From the diagram below, we can see that vector of the linear translation by θ_2 is given by the vector $v_2\theta_2$ where v_2 is the unit vector that gives the direction of the translation and θ_2 is gives the magnitude of the translation.



$$q_1 = q_3 + (-v_2\theta_2)$$

$$v_2\theta_2 = q_3 - q_1 = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ -5 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

To find v_2, we divide $v_2\theta_2$ by its magnitude $\|v_2\theta_2\|$ to get the direction (unit vector).

$$\|v_2\theta_2\| = \sqrt{0 + 4^2 + 4^2} = \sqrt{2 \times 4^2} = \sqrt{2}\sqrt{4^2} = 4\sqrt{2}$$

$$v_2 = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \div 4\sqrt{2} = \begin{bmatrix} 0 \\ 4/_{4\sqrt{2}} \\ 4/_{4\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/_{\sqrt{2}} \\ 1/_{\sqrt{2}} \end{bmatrix}$$

Lastly, we find $J_{b1} = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix}$. This is similar to J_{b3}.

$$\omega_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = -\omega_1 \times q_1 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -5 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ 5 \end{bmatrix}$$

Putting all together,

$$J_b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -7 & 1/\sqrt{2} & -3 \\ 5 & 1/\sqrt{2} & 1 \end{bmatrix}$$