Tutorial 2: Rigid Body Motions

These questions are from the Practice Exercises of the Modern Robotics book. The solutions can be found on the book website. Please try your best before referring to the solutions. You should understand how to solve the problems.

Question 1: Transformation

Let the orientation of {b} relative to {a} be

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

And a point p be represented in {a} as $p_a = (1,2,3)$. What is p_b? Give a numeric 3-vector.

Question 2: Transformation

The mobile manipulator in Figure 1 needs to orient its gripper to grasp the block. For subsequent placement of the block, we have decided that the orientation of the gripper relative to the block, when the gripper grasps the block, should be R_{eg}. Our job is to determine the rotation operator to apply to the gripper to achieve this orientation relative to the block. Figure 1 shows the fixed world frame {a}, the mobile robot's chassis frame {b}, the gripper frame {c} (this is confusing, assume there is a fixed transformation from {c} to the actual gripper frame {g}, R_{cg}), the RGBD camera (color vision plus depth, like the Kinect) frame {d}, and the object frame {e}. Because we put the camera at a known location in space, we know R_{ad}. The camera reports the configuration of {e} relative to {d}, so we know R_{de}. From the mobile robot's localization procedure (e.g., vision-based localization or odometry) we know R_{ab}. From the robot arm's forward kinematics we know R_{bc}.

a. In terms of the four known rotation matrices R_{ad}, R_{de}, R_{ab}, and R_{bc}, and using only matrix multiplication and the transpose operation, express the current orientation of the gripper relative to the block, R_{ec}.
b. To align the gripper properly, you could apply to it a rotation $R1$ expressed in terms of axes in the gripper's {c} frame. What is $R1$, in terms of the five known rotation matrices (R_{ad}, R_{de}, R_{ab}, R_{bc}, R_{eg}), matrix multiplication, and transpose?

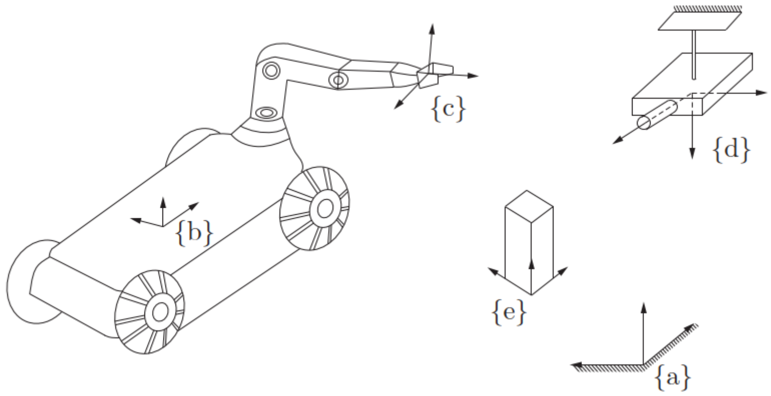

Figure 1 The fixed world frame {a}, the mobile robot's chassis frame {b}, the gripper frame {c}, the RGBD camera frame {d}, and the object frame {e}.

Question 3: Twist

Figure 2 shows a screw, a frame {b}, and a frame {s}. The $\hat{x}_b$ -axis of {b} is along the axis of the screw, and the origin of the frame {s} is displaced by 2 cm, along the $\hat{y}_b$ -axis, from the {b} frame. The $\hat{z}_s$ -axis is aligned with $\hat{x}_b$ and the $\hat{x}_s$ -axis is aligned with $\hat{z}_b$.

Taking note of the direction of the screw's threads, as the machine screw goes into a tapped hole driven by a screwdriver rotating at 3 radians per second with a pitch of 4π mm per revolution (this incur linear velocity), what is the screw's twist expressed in {b}, $\mathcal{V}_b$? What is the screw axis expressed in {b}, $\mathcal{S}_b$? What is $\mathcal{V}_s$? What is $\mathcal{S}_s$? Give units as appropriate.

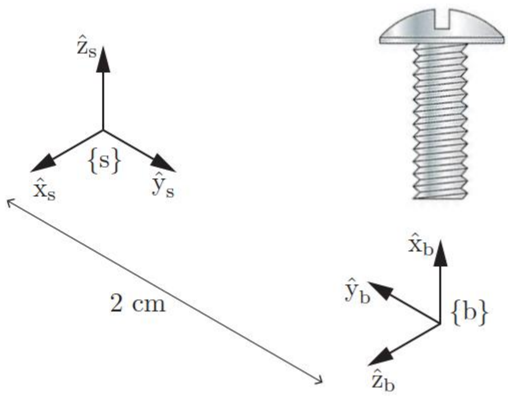

Figure 2 As the machine screw goes into a tapped hole, it advances linearly by 4π mm every full rotation of the screw.

Sample solution

Ref: Modern Robotics Practice Exercises

Question 1: Transformation

Note $R = R_{ab}$ and p_a is the point p expressed in {a}. Both R and p_a are expressed in the same frame {a}. Now we want to express the point p in {b}, so we need to transform from {a} to {b}, i.e. we need to multiply p_a by the transformation from {a} to {b}, i.e. the orientation of {a} in {b}, R_{ba}.

$$p_b = R_{ba} p_a = {R_{ab}}^T p_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = (1,3,-2)$$

Question 2: Transformation

(b) {g} and {c} frames are confusing. {g} could be omitted here. For this answer, it is assumed there is an additional gripper frame {g} besides the {c}.

Solution 3.1

(a)

$$\begin{aligned} R_{ec} &= R_{ed} R_{da} R_{ab} R_{bc} \\ &= R_{de}^{\mathrm{T}} R_{ad}^{\mathrm{T}} R_{ab} R_{bc}. \end{aligned}$$

(b)

$$\begin{aligned} R_{ec} R_1 = R_{eg} \rightarrow R_1 &= R_{ec}^{\mathrm{T}} R_{eg} \\ &= (R_{de}^{\mathrm{T}} R_{ad}^{\mathrm{T}} R_{ab} R_{bc})^{\mathrm{T}} R_{eg} \\ &= R_{bc}^{\mathrm{T}} R_{ab}^{\mathrm{T}} R_{ad} R_{de} R_{eg} \quad (= R_{cg}). \end{aligned}$$

(c)

$$R_2 = R_{bc} R_1 = R_{bc} R_{cg} = R_{bc} R_{bc}^{\mathrm{T}} R_{ab}^{\mathrm{T}} R_{ad} R_{de} R_{eg} = R_{ab}^{\mathrm{T}} R_{ad} R_{de} R_{eg} \quad (= R_{bg}).$$

Question 3: Twist

$\mathcal{V}_b = (\omega_b, v_b) = (-3,0,0,-0.006,0,0)$ where

Screw rotate in the direction of $-\hat{x}_b$, $\omega_b = -\hat{x}_b \times \dot{\theta} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} . 3 = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$ and

$$v_b = -\omega_b \times (q = \hat{x}_b) + h\omega_b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -0.006 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.006 \\ 0 \\ 0 \end{bmatrix}$$

Note $h = 4\pi \; mm \; per \; revolution \; \times 3 \frac{rad}{s} = 2 \; mm \; per \; rad \; \times 3 \frac{rad}{s} = 6 \frac{mm}{s} = 0.006 \; m/s.$

$$\mathcal{V}_b = \mathcal{S}_b\dot{\theta} \text{ where}$$

$$\dot{\theta} = \|\omega_b\| = 3, \text{ so } \mathcal{S}_b = \frac{\mathcal{V}_b}{\dot{\theta}} = (-1,0,0,-0.002,0,0)$$

$\mathcal{V}_s = (\omega_s, v_s) = (0,0,-3,-0.06,0,-0.006)$ where

Screw rotate in the direction of $-\hat{z}_s$, $\omega_s = -\hat{z}_s \times \dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.3 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ and

$$v_s = -\omega_s \times (q = \hat{y}_s.0.02) + h\omega_s = -\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0.02 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.006 \end{bmatrix} = \begin{bmatrix} -0.06 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.006 \end{bmatrix}$$
$$= \begin{bmatrix} -0.06 \\ 0 \\ -0.006 \end{bmatrix}$$

Note that the rotating direction of ω_s (in $-\hat{z}_s$) and the distance of $\{s\}$ from point q (at $\{b\}$), we could visualize that the first component (in yellow) of v_s is in $-\hat{x}_s$ direction (right hand rule rotation around the screw) similar to what we did for getting the screw axis in forward kinematic problem. Doing so, we obtain $v_s = \begin{bmatrix} -0.02 \\ 0 \\ 0 \end{bmatrix} + h\omega_s$, which is incorrect. This is because in FK case, we look for the cross product (orthogonal vector) to a unit axis $\widehat{\omega}_s$ and a vector q, whereas in here ω_s is not a unit vector, it should be scaled by $\dot{\theta}$, i.e. $\omega_s = \widehat{\omega}_s\dot{\theta}$. In this simple case, we can simply do $v_s = \begin{bmatrix} -0.02 \\ 0 \\ 0 \end{bmatrix}\dot{\theta} = \begin{bmatrix} -0.06 \\ 0 \\ 0 \end{bmatrix}$, however this will not work if ω_s or q has more nonzero elements. It is safer to perform a cross product to obtain v_s.

$$\mathcal{V}_s = \mathcal{S}_s\dot{\theta} \text{ where}$$

$$\dot{\theta} = \|\omega_s\| = 3, \text{ so } \mathcal{S}_s = \frac{\mathcal{V}_s}{\dot{\theta}} = (0,0,-1,-0.02,0,-0.002)$$

Solution 3.2 The threads of this screw are the typical right-handed threads, which means that the screw, when viewed from the top, rotates clockwise when it advances into a tapped hole. In other words, the fingers of your right hand curl in the direction of rotation of the screw when your right thumb points downward on the page, in the negative direction of the upward-pointing $\hat{\mathrm{x}}_\mathrm{b}$-axis. Since the screwdriver rotates at 3 rad/s, the screw also rotates at 3 rad/s, so the angular component of the twist, expressed in {b}, is $\omega_b = (-3 \text{ rad/s}, 0, 0)$. Since radians and seconds are the SI units for angle and time, respectively, you could write $(-3, 0, 0)$ and assume the default SI units. You could also write $(-3(180/\pi) \text{ deg/s}, 0, 0)$, but that would be unusual.

The pitch of the screw is 4π mm per revolution, or 2 mm/rad. So as the screw is rotated at 3 rad/s, it moves linearly in the $-\hat{\mathrm{x}}_\mathrm{b}$ direction at (2 mm/rad)(3 rad/s) = 6 mm/s. So the linear component of the twist expressed in {b} is $(-6 \text{ mm/s}, 0, 0)$, or, in SI units, $v_b = (-0.006 \text{ m/s}, 0, 0)$. So, in SI units, the entire twist is $\mathcal{V}_b = (\omega_b, v_b) = (-3, 0, 0, -0.006, 0, 0)$.

The corresponding screw axis expressed in {b} is the normalized version of $\mathcal{V}_b$ where the magnitude of the angular velocity is unit. The magnitude of ω_b is 3, so divide the twist by 3 to get $\mathcal{S}_b = (-1, 0, 0, -0.002, 0, 0)$. We can write $\mathcal{V}_b = \mathcal{S}_b\dot{\theta}$ where $\dot{\theta} = \|\omega_b\| = 3$.

The screw axis could also be represented in the {b} frame by the collection $\{q_b, \hat{s}_b, h\}$, where a point q_b on the axis is $(0, 0, 0)$ (expressed in {b}), the axis direction is $\hat{s}_b = (-1, 0, 0)$, and the pitch is $h = 0.002$.

In the {s} frame, the axis of rotation is aligned with the $-\hat{\mathrm{z}}_\mathrm{s}$-axis, so $\omega_s = (0, 0, -3)$. A point at the origin of {s}, rigidly attached to the advancing screw, has a downward linear component of -0.006 m/s in the $-\hat{\mathrm{z}}_\mathrm{s}$ direction (i.e., $(0, 0, -0.006)$) from the downward motion of the screw. But it also has a linear component in the $-\hat{\mathrm{x}}_\mathrm{s}$ direction from the rotation of the screw. The point at the origin of {s} can be expressed as $q_b = (0, 0.02, 0)$ in terms of {b} coordinates, so the linear motion at {s} due to the rotation of the screw is $\omega_b \times q_b = (0, 0, -0.06)$. In the {s} frame, this is $(-0.06, 0, 0)$. (Imagine a turntable rotating about the screw axis and the resulting motion of a point at {s}.) So the total linear motion at {s}, expressed in {s}, is $v_s = (0, 0, -0.006) + (-0.06, 0, 0) = (-0.06, 0, -0.006)$. Therefore, $\mathcal{V}_s = (0, 0, -3, -0.06, 0, -0.006)$. The screw axis is $\mathcal{S}_s = (0, 0, -1, -0.02, 0, -0.002)$ and $\mathcal{V}_s = \mathcal{S}_s\dot{\theta}$.

The screw axis could also be represented in the {s} frame by the collection $\{q_s, \hat{s}_s, h\}$, where a point q_s on the axis is $(0, 0.02, 0)$, the axis direction is $\hat{s}_s = (0, 0, -1)$, and the pitch is $h = 0.002$. Note that $\mathcal{S}_s = (\hat{s}_s, -\hat{s}_s \times q_s + h\hat{s})$, where $h\hat{s}$ is the linear velocity due to the linear motion of the screw and $-\hat{s}_s \times q_s$ is the linear velocity due to the rotation of the screw.

You could also calculate $\mathcal{V}_s$ and $\mathcal{S}_s$ using $\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$ and $\mathcal{S}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{S}_b$.