Forward Kinematics: Manipulators

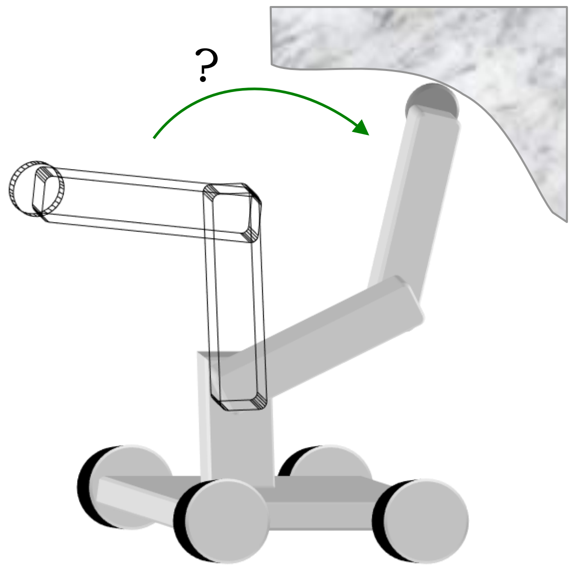
ZA-2203 Robotic Systems

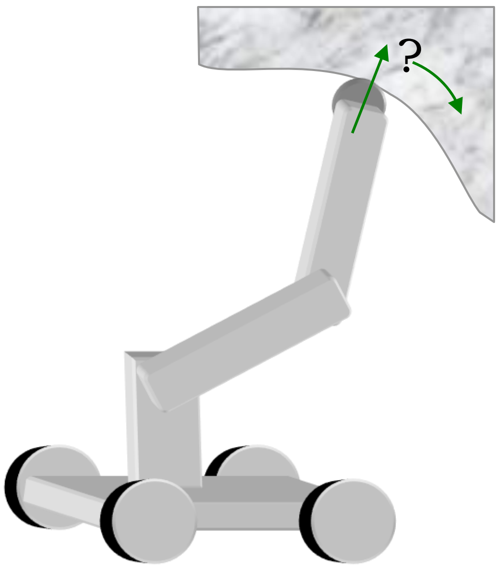
Topics

- Kinematics vs dynamics
- Forward kinematics vs inverse kinematics
- Forward kinematics computation
- Power of Exponential (PoE) to compute forward kinematics in base form
- PoE in body form

Two aspects of motions

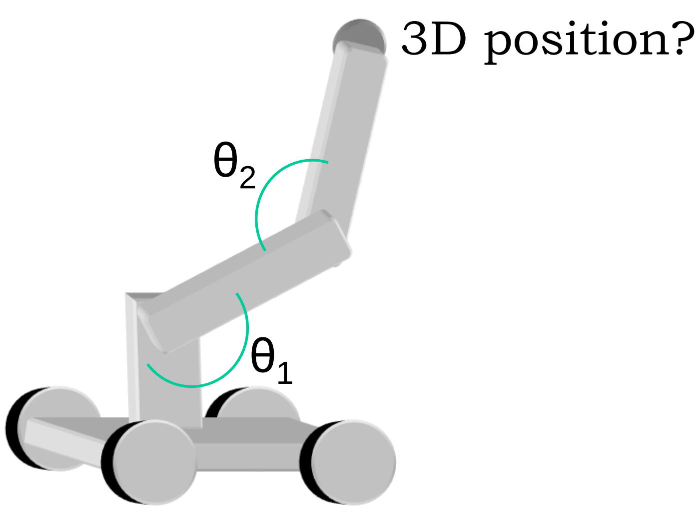
- **Kinematics** – is the study of motion without regard to forces.
  - Study of correspondence between actuator mechanisms and the resulting motion of effectors.
- **Kinetics** – also called **Dynamics**, is the study of motion with regards to forces

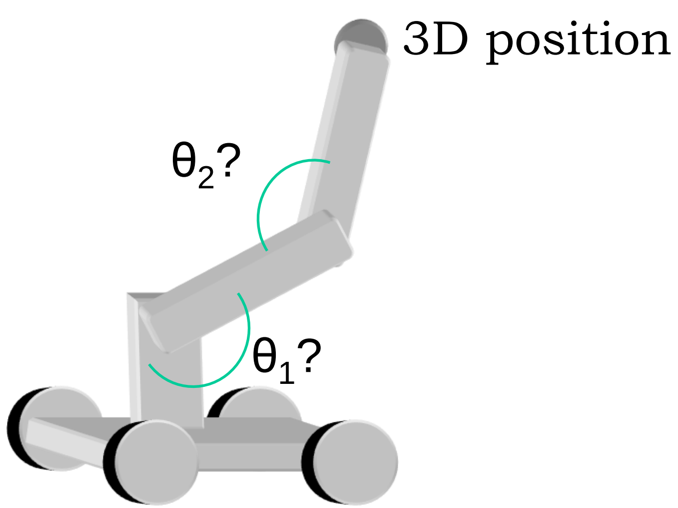




Forward & Inverse Kinematics

- **Kinematics** – is the study of motion without regard to forces.
  - Study of correspondence between actuator **mechanisms** and resulting **motion** of effectors.
- **Forward Kinematics** (FK): for the given angular movement at each joint, where will the end-effector reach?
- **Inverse Kinematics** (IK): for a desired position of the end-effector, how much should each joint rotate?



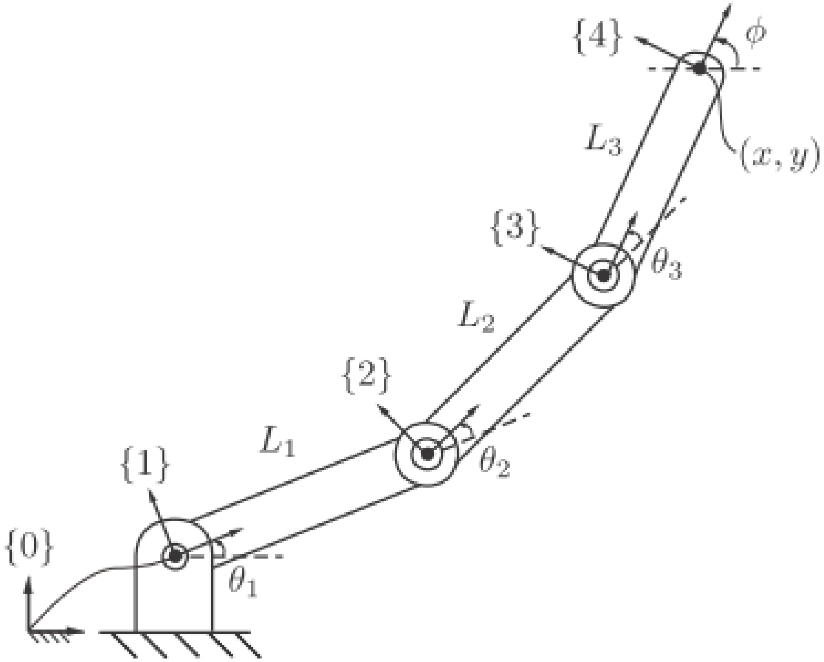


Forward kinematics: three approaches

- Three approaches for FK:
  - Geometry of the mechanisms
  - Homogeneous transformations
  - Power of exponentials

3R planar open chain manipulator: geometry

FK: Determine Pose of {4} in {0} given $\theta_1, \theta_2,\ \theta_3$
IK: Determine $\theta_1, \theta_2,\ \theta_3$ given pose of {4} in {0}

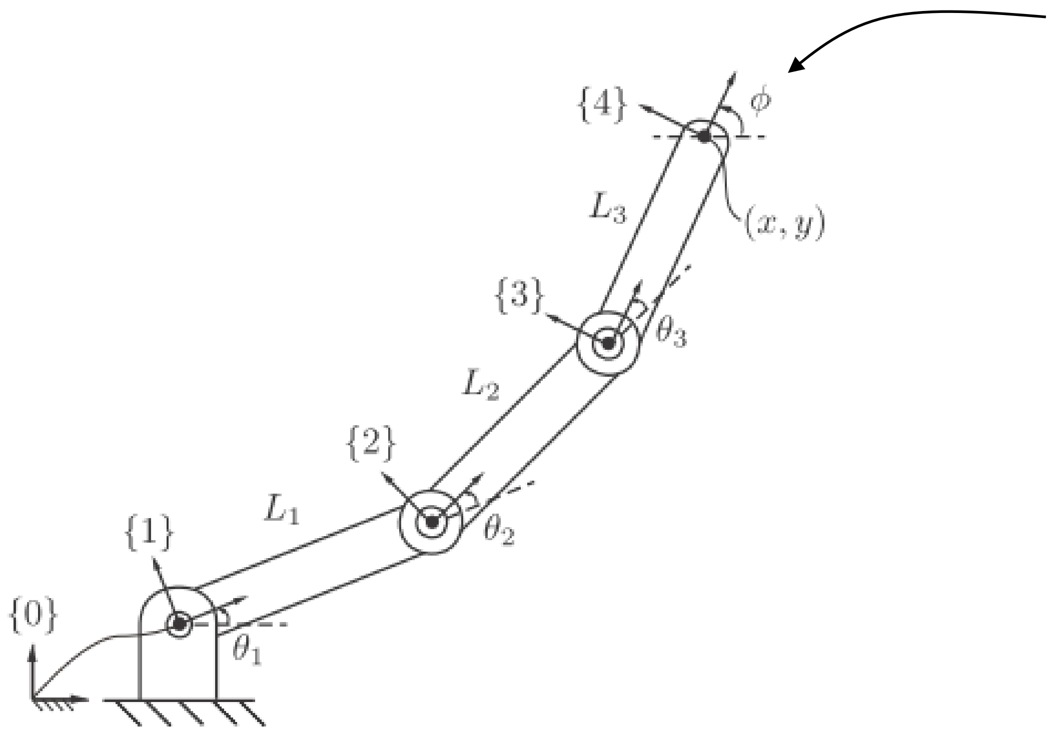


Source: Modern Robotics

3R planar open chain manipulator: geometry

FK: Determine Pose of {4} in {0} given $\theta_1, \theta_2, \theta_3$

x, y, ϕ ?



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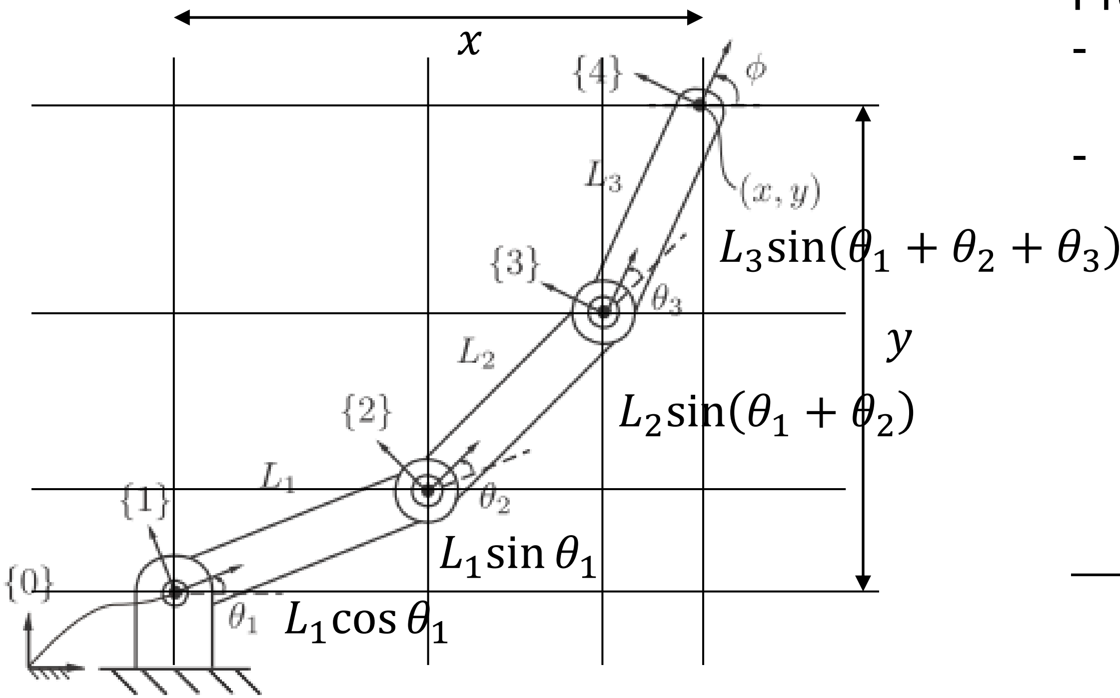
3R planar open chain manipulator: geometry

FK: Determine Pose of {4} in {0} given $\theta_1, \theta_2, \theta_3$



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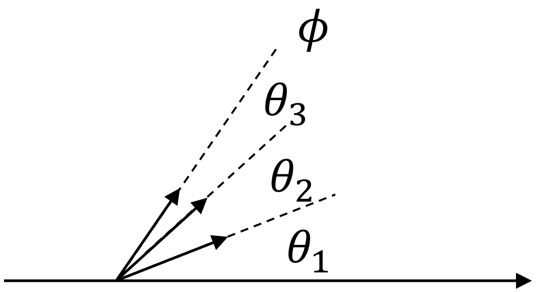
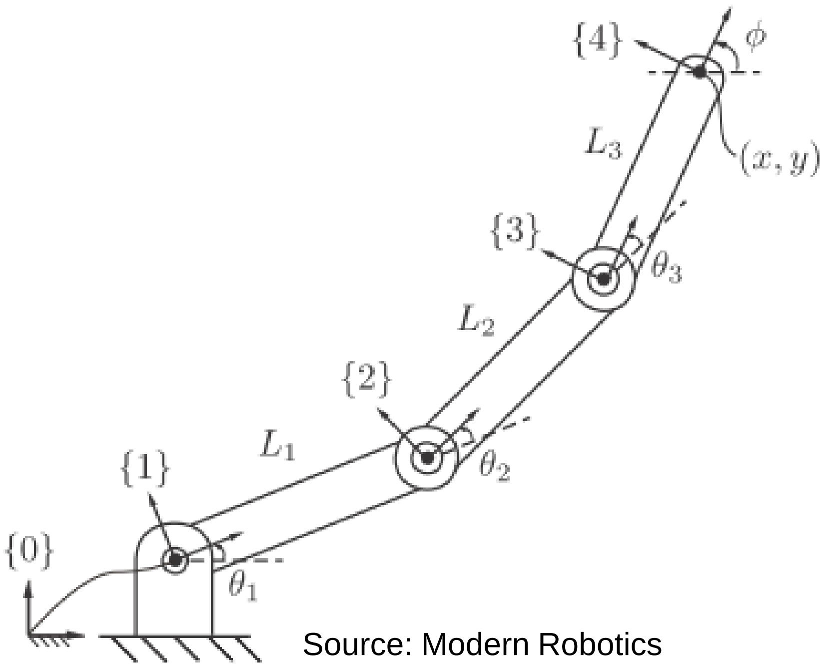
From **geometry**.
- Not easy for more complex manipulator
- Not systematic



$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3), \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3), \\
\phi &= \theta_1 + \theta_2 + \theta_3.
\end{aligned}
$$

A 3R planar open chain manipulator: T matrix

$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$



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$$T_{01} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

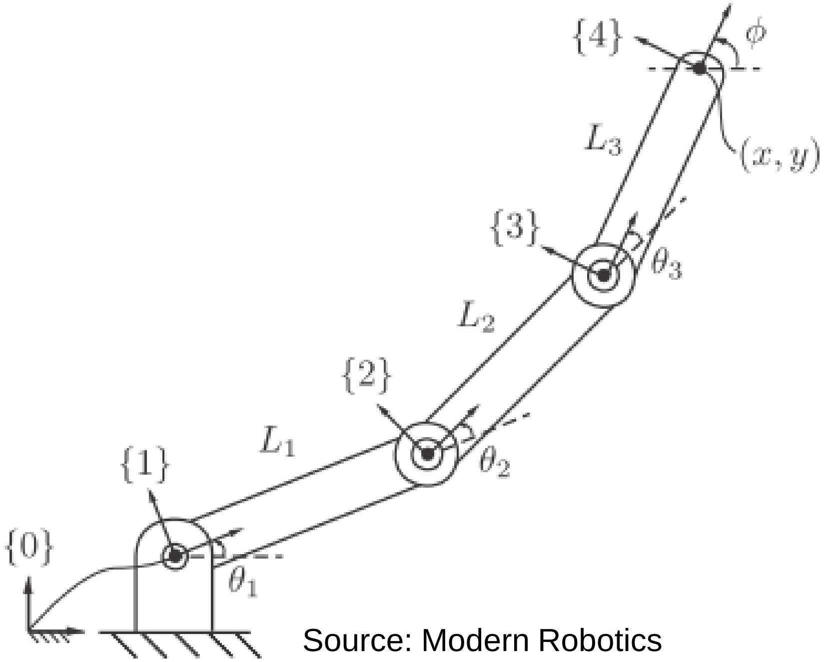
$$T_{34} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Attach coordinate frame to each link (body) and use **transformation matrix**.

- Requires correctly attach coordinate frame to all links
- The approach used by **Denavit-Hartenberg (DH) convention**
- For n joints, has 3n parameters, and n joint variables

A 3R planar open chain manipulator: T matrix

$$T_{04} = T_{01}T_{12}T_{23}T_{34}$$



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$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Attach coordinate frame to each link (body) and use **transformation matrix**.

- Requires correctly attach coordinate frame to all links
- The approach used by **Denavit-Hartenberg (DH) convention**
- For n joints, has 3n parameters, and n joint variables

On screw motion

Recall **transformation matrix** represents rotation and translation

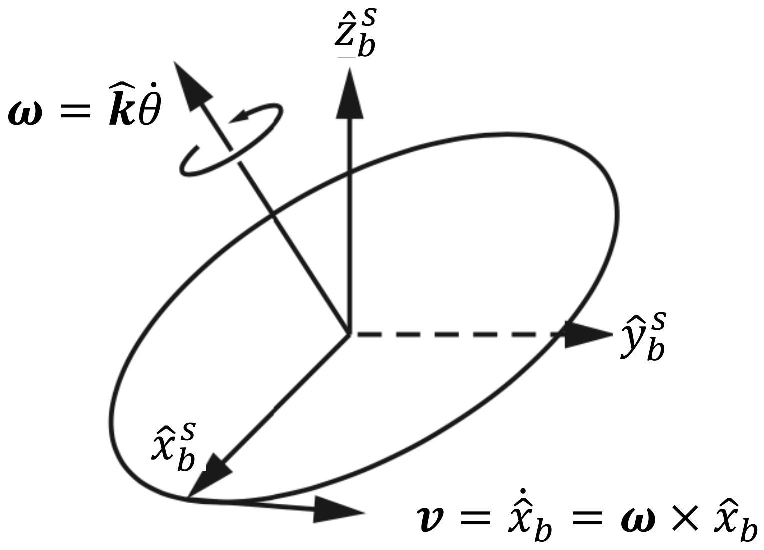
$$transformation\ T = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

A transformation can also be represented by a rotation around a **screw axis**

$$\omega\ is\ the\ angular\ velocity, v\ is\ the\ linear\ velocity$$

A **twist** encapsulates the angular velocity and linear velocity in one data structure

$$\mathcal{V} = (\omega, v)$$



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On screw motion

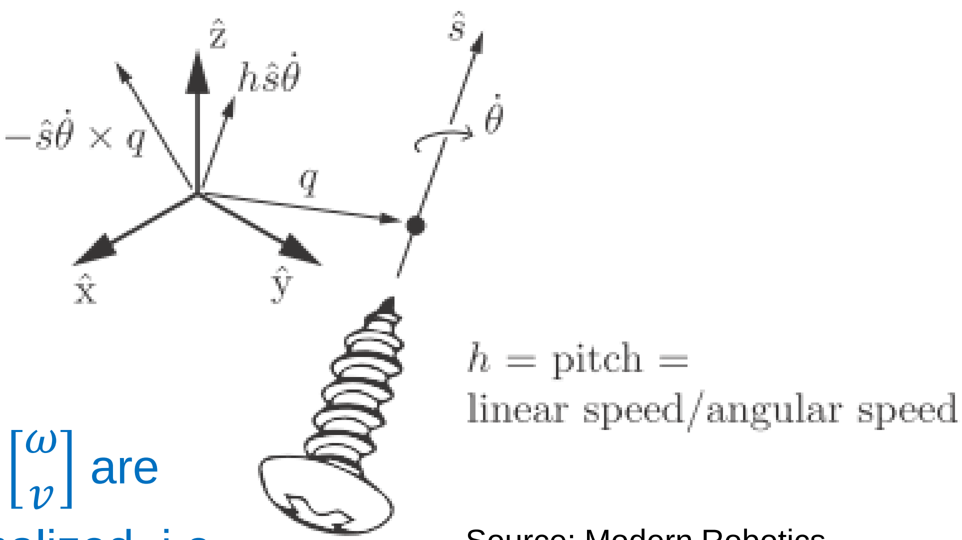
We can think of the **twist** as coming from a unit **screw axis** $\mathcal{S}$ (direction) rotating at an **angular speed** of $\dot{\theta}$.

$$Twist\ \mathcal{V} = \mathcal{S}\dot{\theta}$$

In this case, we set $\|\omega\| = 1$ (motion involves rotation), or $\|\omega\| = 0$ and $\|v\| = 1$ (motion has no rotation, only translation), then we have

$$Screw\ axis\ \mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

In the case $\|\omega\| = 1$, $\dot{\theta}$ is the angular velocity about the screw axis. In the case $\|\omega\| = 0$ and $\|v\| = 1$, $\dot{\theta}$ is the linear velocity along the screw axis.



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Confusion: ω, v in $\mathcal{V} = (\omega, v)$ and $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix}$ are different. The ω, v in $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix}$ are normalized, i.e. without $\dot{\theta}$.

On screw motion

Screw axis can be written in matrix form

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \qquad [\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

This screw axis can be used in **matrix exponent** to obtain the **transformation matrix**

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix},$$

We can use **Rodrigues' formula** to expand the matrix exponent

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

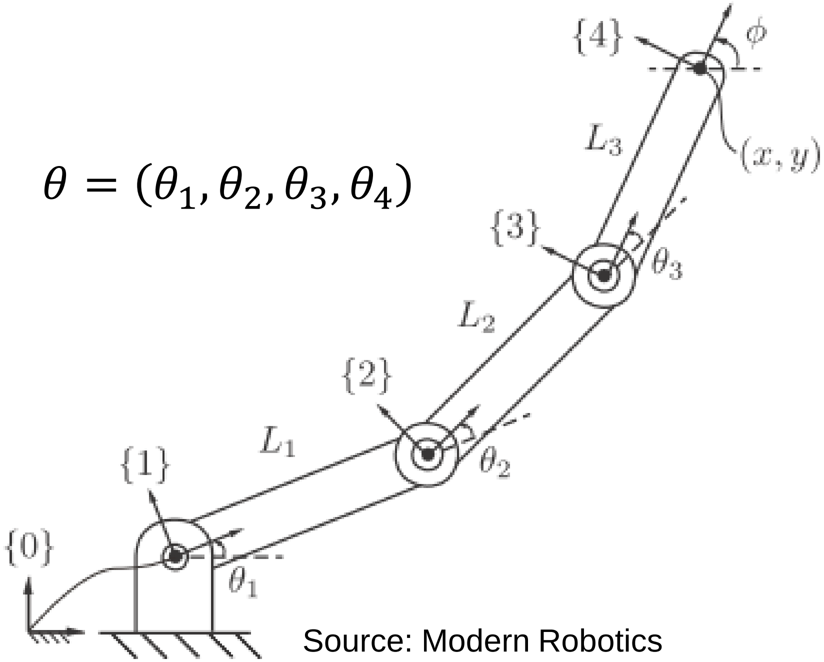
Or

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v \\ 0 & 1 \end{bmatrix}$$

A 3R planar open chain manipulator: screws

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$



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T_{01} (rotated by θ_1)

T_{02} (rotated by θ_2)

T_{03} (rotated by θ_3)

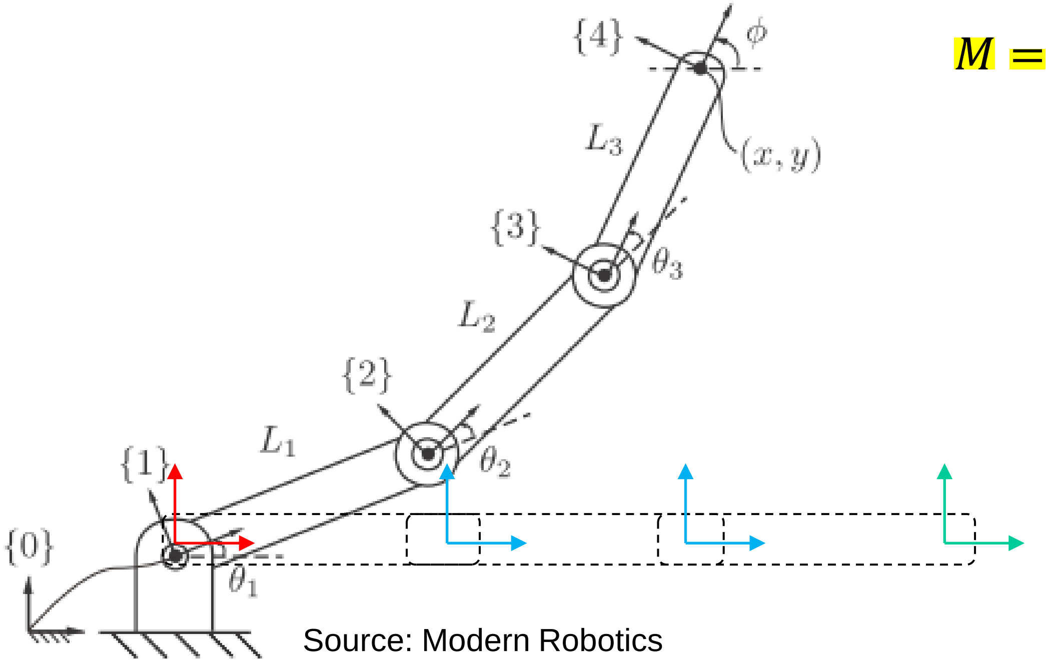
$T_{04\,(\theta=0)} = T(0)$

Reference to fix frame, hence pre-multiply

Use **matrix exponential** with screw axes

- Assign a screw axis for each link, at the joint
- The approach used by **Product of Exponentials (PoE)**
- For n joints, requires 6n parameters and n joint variables

A 3R planar open chain manipulator: screws



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Pose at 0 (zero), e.g. all angles 0

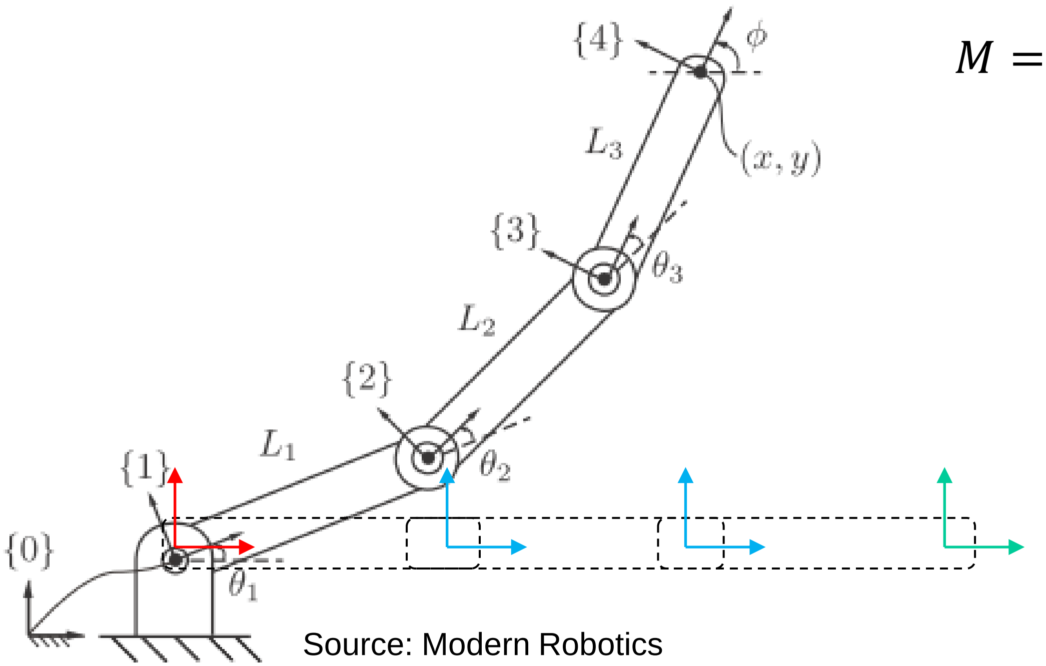
$$M = T(0) = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

A 3R planar open chain manipulator: screws

Pose at 0 (zero), e.g. all angles 0

$$M = T(0) = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



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$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

A 3R planar open chain manipulator: screws

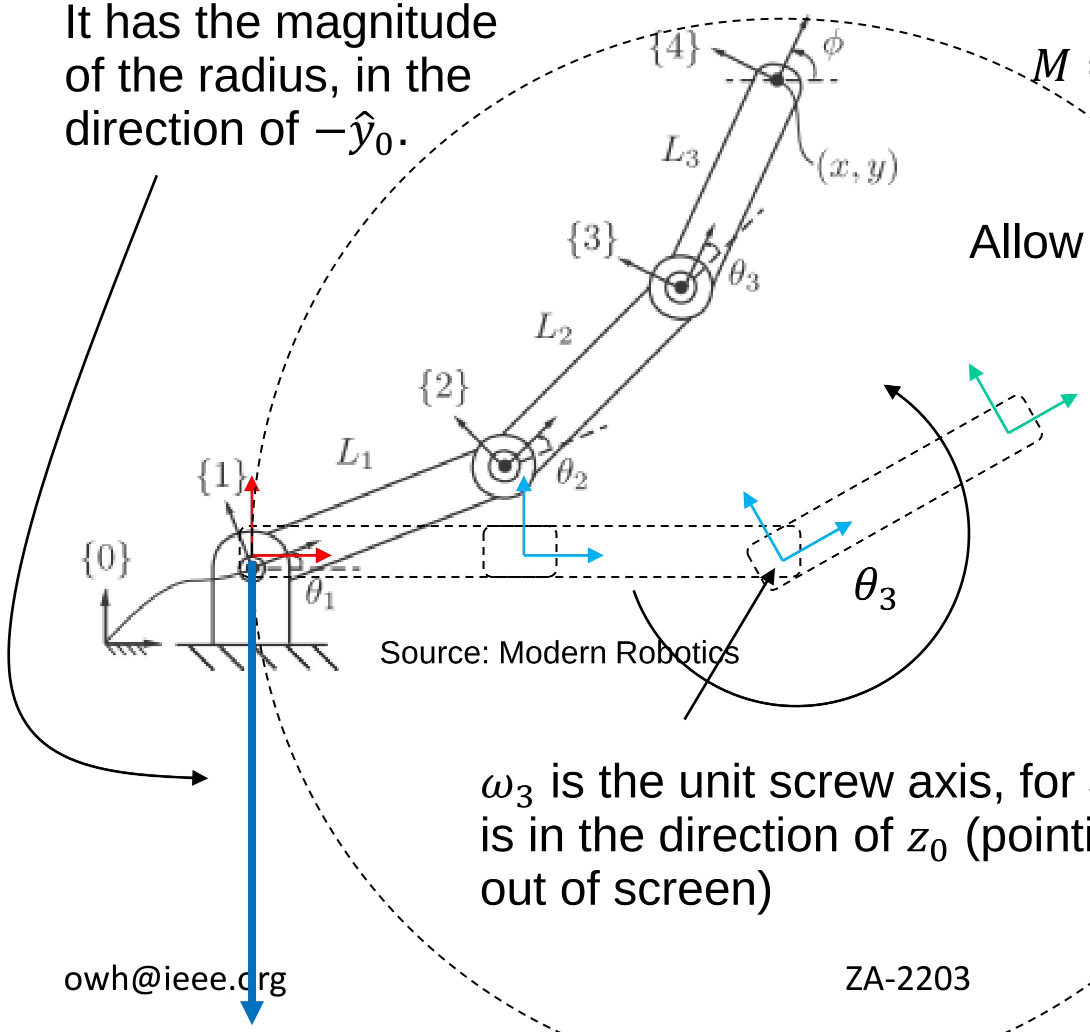
v_3 is the linear velocity of a point at {0} on the rotation circle around J3.
It is the perpendicular vector at {0}.
It has the magnitude of the radius, in the direction of $-\hat{y}_0$.

Pose at 0 (zero), e.g. all angles 0

$$M = T(0) = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Allow rotate about J3, fix J1 and J2 at 0

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$



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ω_3 is the unit screw axis, for J3, it is in the direction of z_0 (pointing out of screen)

A 3R planar open chain manipulator: screws

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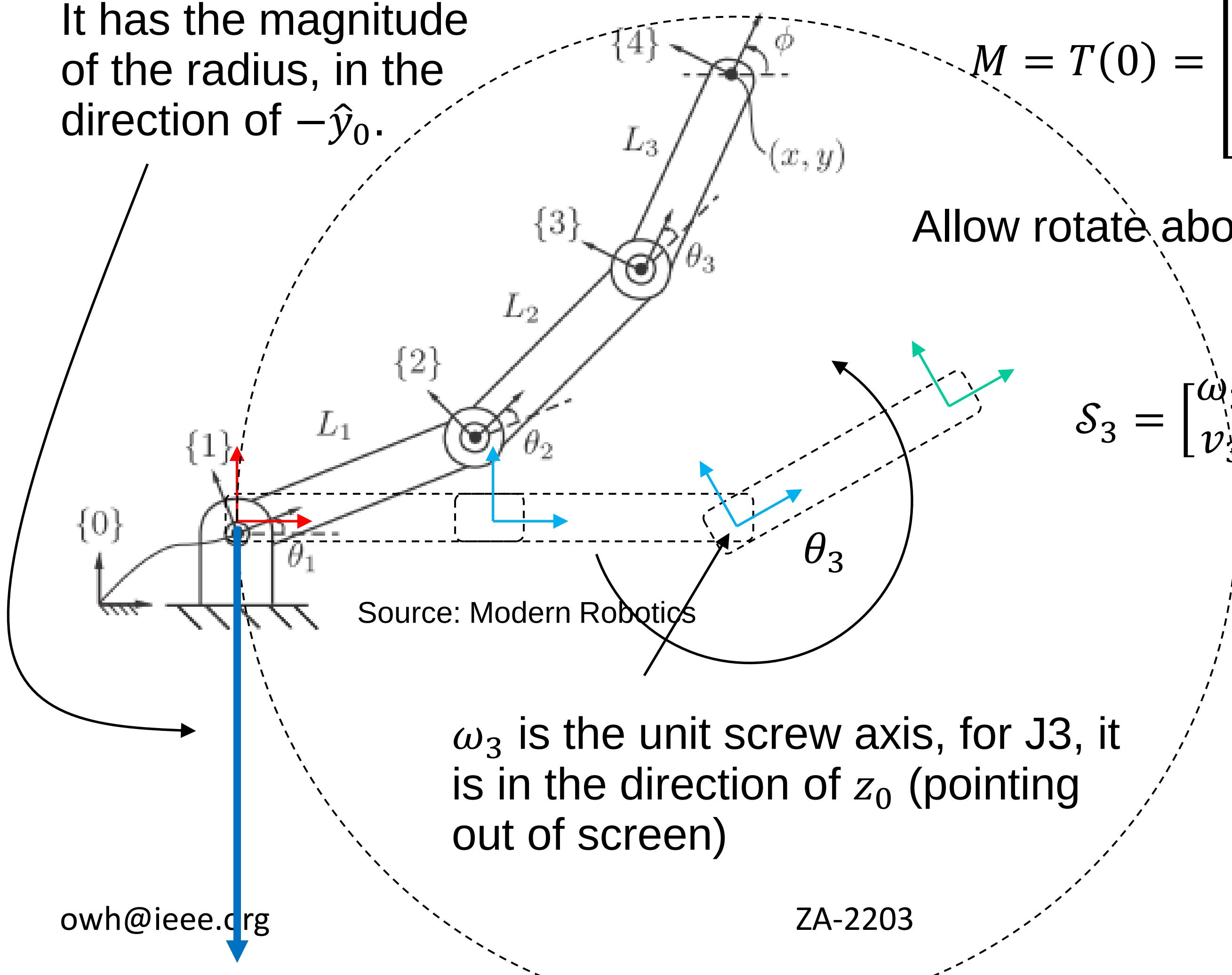
Allow rotate about J3, fix J1 and J2 at 0

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$



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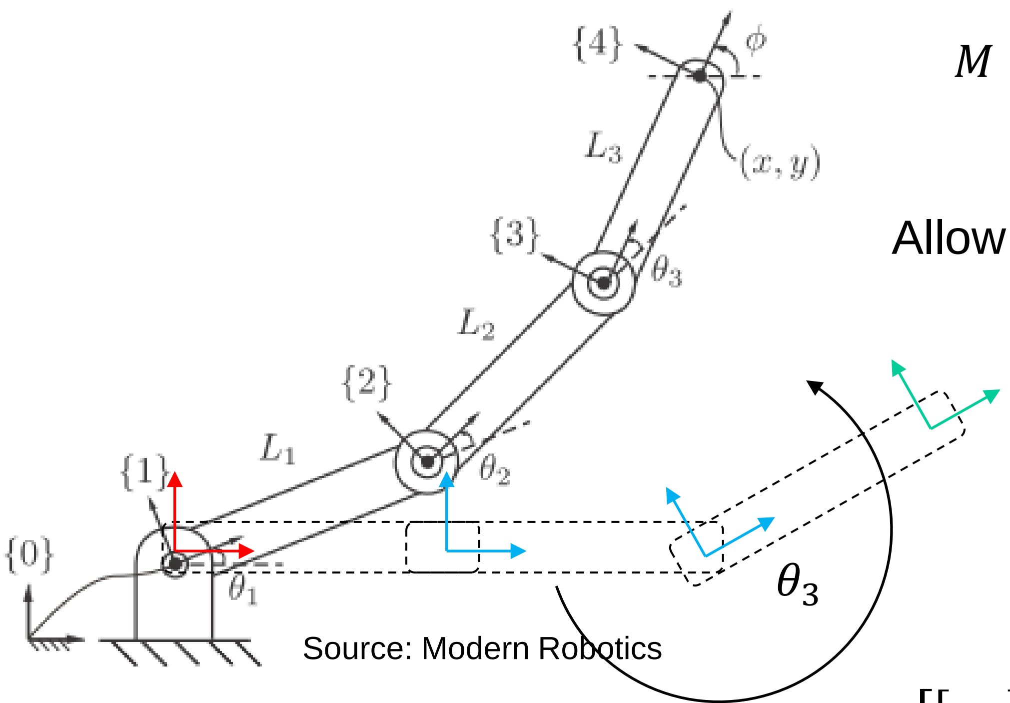
A 3R planar open chain manipulator: screws



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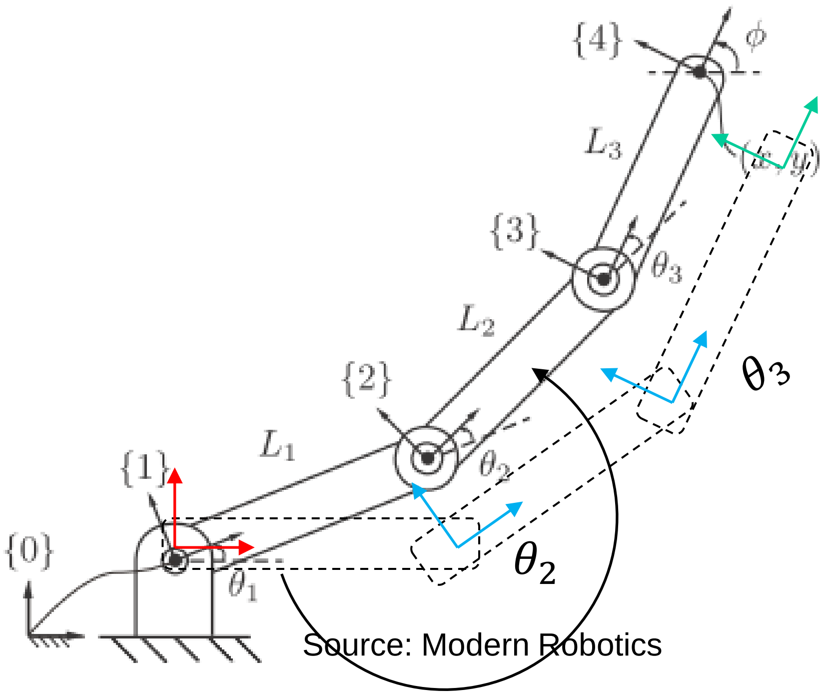
$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

$T(\theta) = e^{[\mathcal{S}_3]\theta_3} M$ if $\theta_1 = \theta_2 = 0$

A 3R planar open chain manipulator: screws



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Pose at 0 (zero), e.g. all angles 0

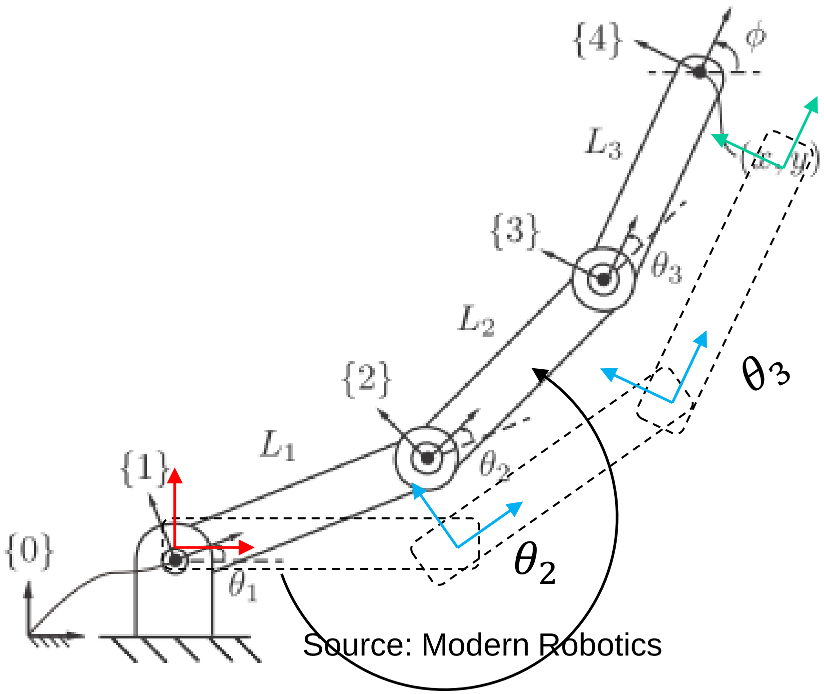
$$M = T(0) = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Allow rotate about J2, fix J1 at 0, J3 fix at θ_3 :

$$\mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

A 3R planar open chain manipulator: screws



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$$T(\theta) = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \text{ if } \theta_1 = 0$$

Pose at 0 (zero), e.g. all angles 0

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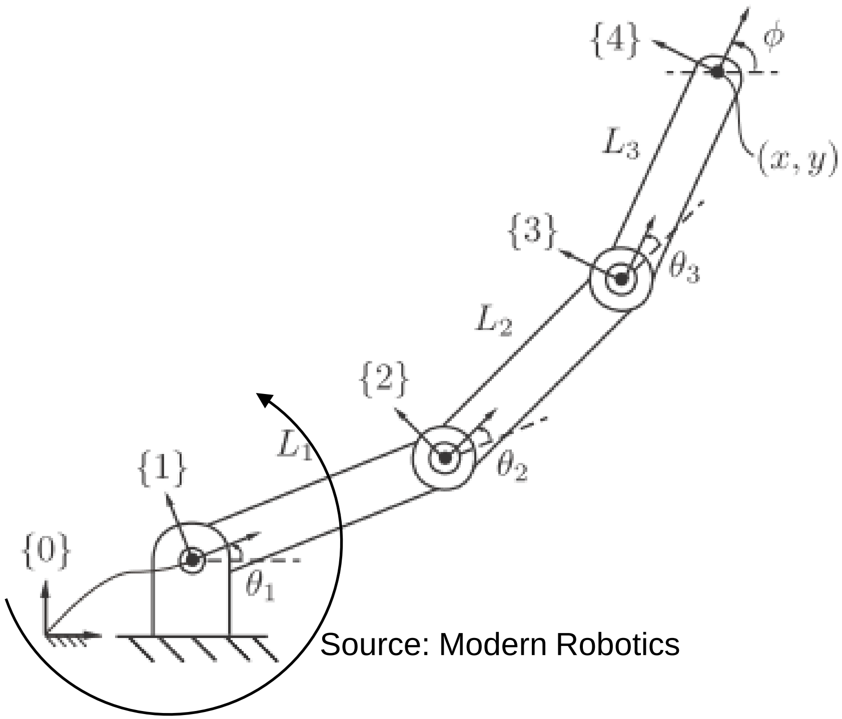
Allow rotate about J2, fix J1 at 0, J3 fix at θ_3 :

$$\mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A 3R planar open chain manipulator: screws



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$$T(\theta) = T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Pose at 0 (zero), e.g. all angles 0

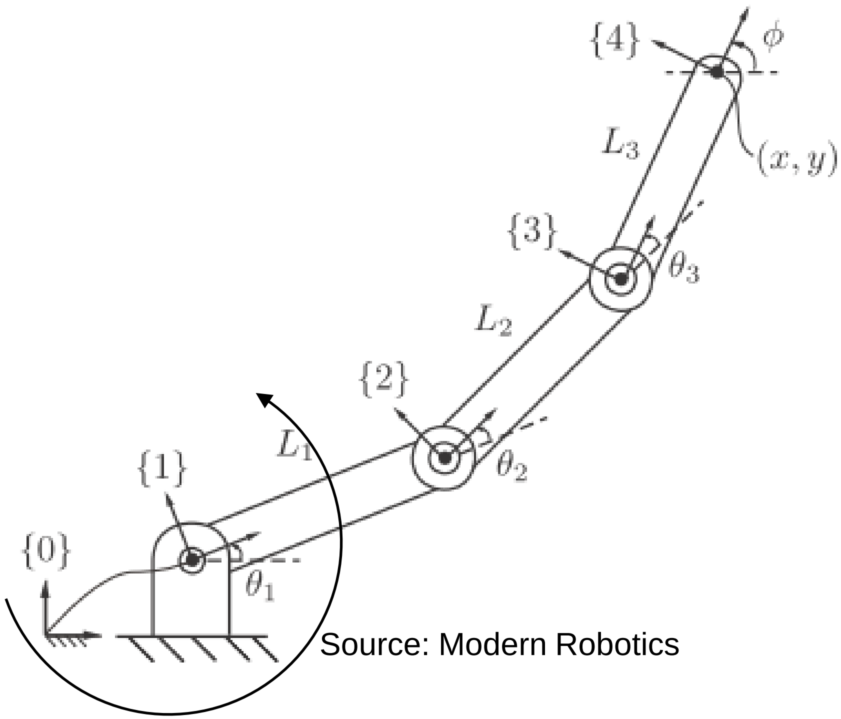
$$M = T(0) = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Allow rotate about J1, fix J2 at θ_2, J3 at θ_3 :

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$[\mathcal{S}_1] = \begin{bmatrix} [\omega_1] & v_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

A 3R planar open chain manipulator: screws



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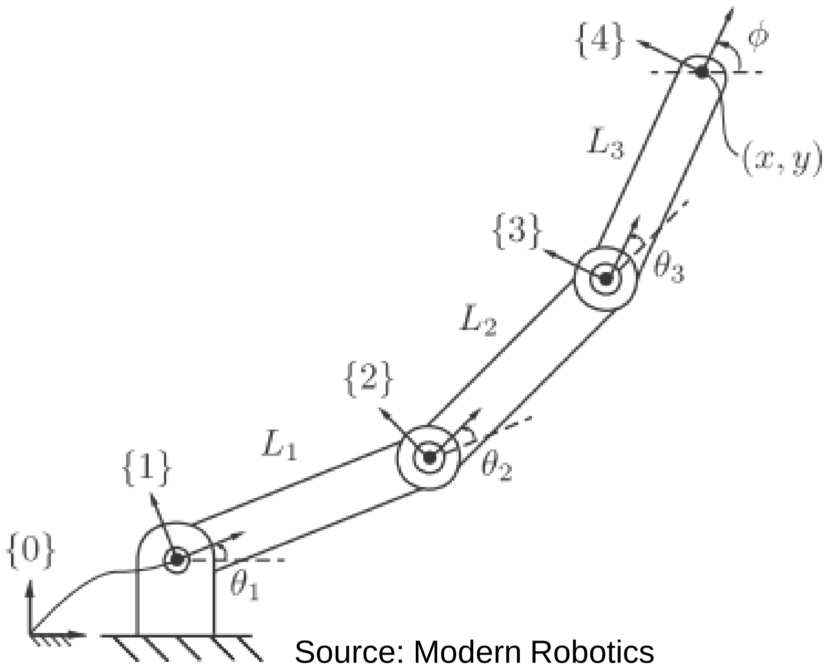
A 3R planar open chain manipulator: screws



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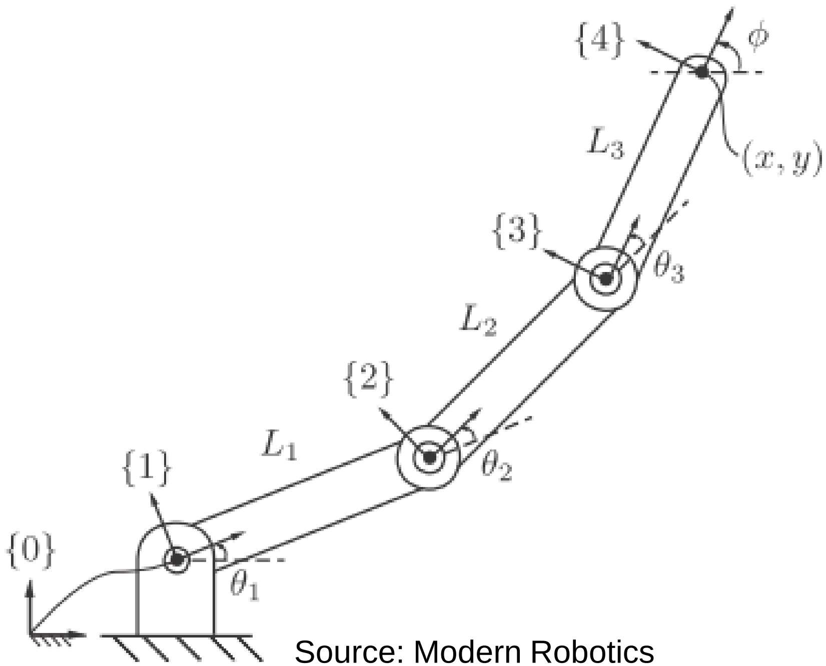
$$[\mathcal{S}_3] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathcal{S}] = [\mathcal{S}_1 \quad \mathcal{S}_2 \quad \mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -L_1 & -(L_1 + L_2) \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

A 3R planar open chain manipulator: screws



Source: Modern Robotics

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,0,1)	$(0, -L_1, 0)$
3	(0,0,1)	$(0, -(L_1 + L_2), 0)$

$$M = T(0) = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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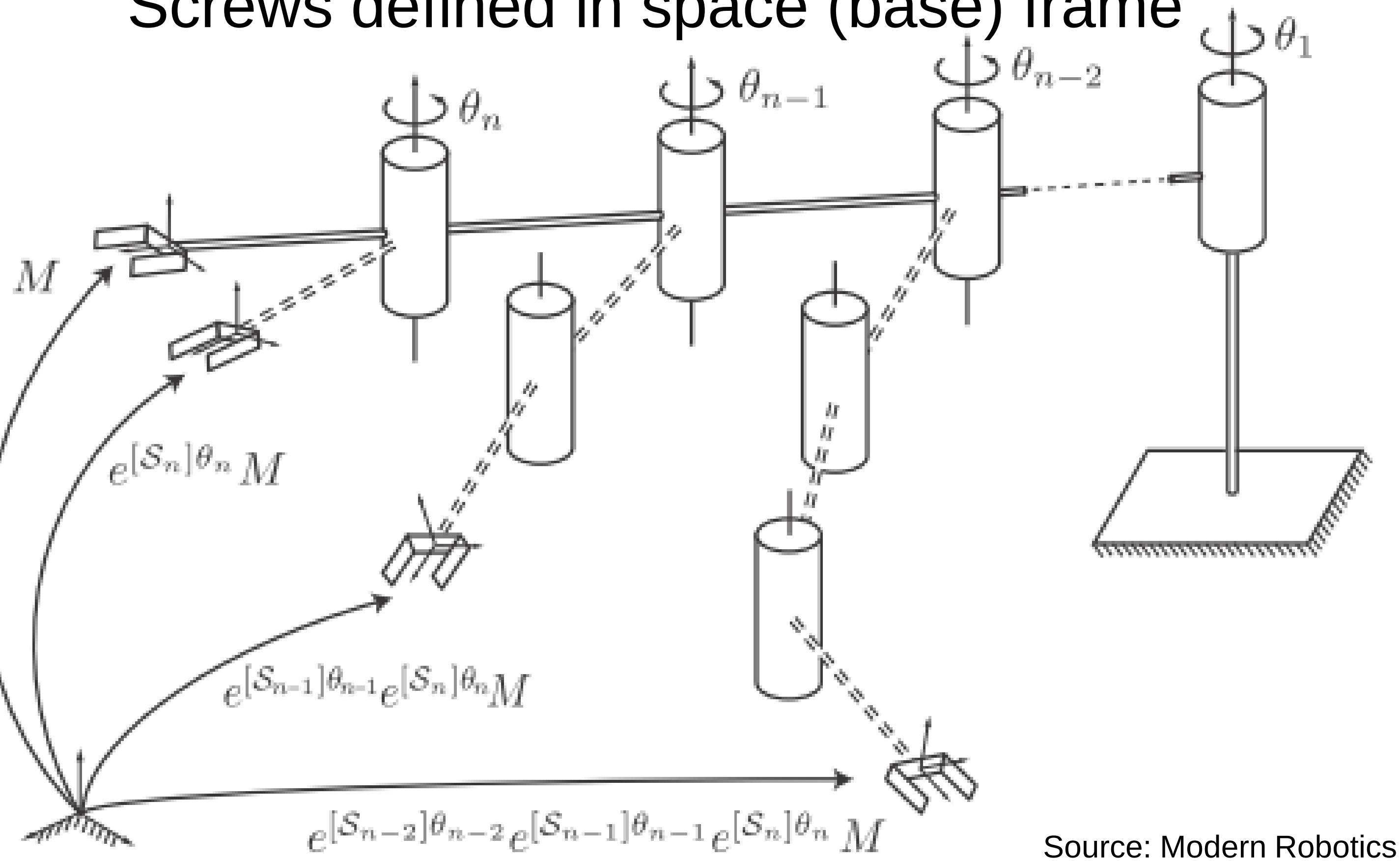
$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

Power of Exponentials (PoE): space form

Screws defined in space (base) frame



Source: Modern Robotics

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

$\mathcal{S}_i$ is defined in space (base) frame {s} or {0}

PoE space form: screws w.r.t. space frame

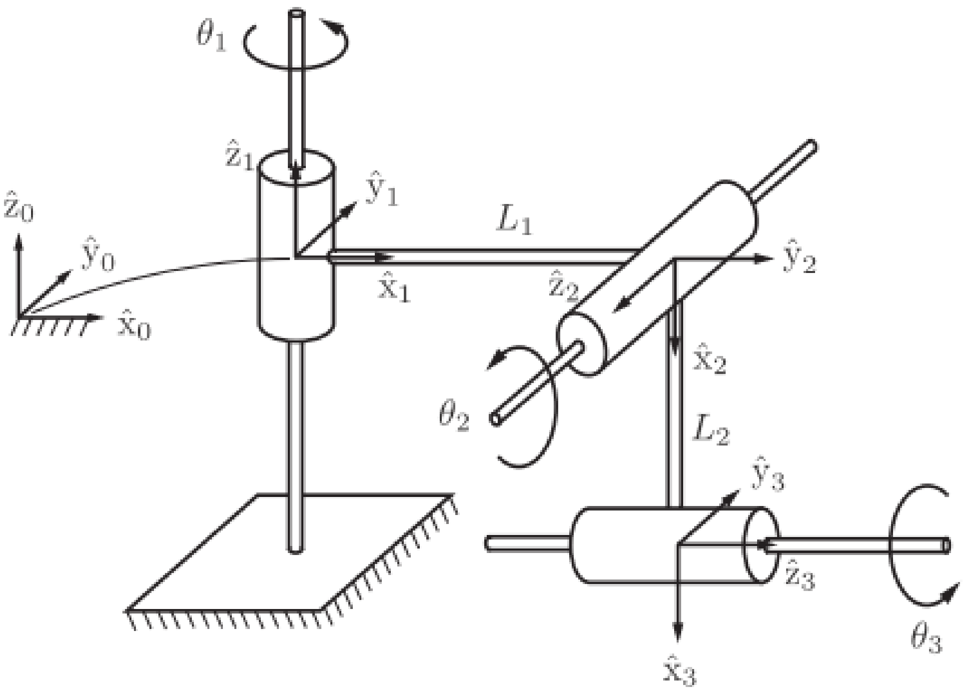
Given

- $M = T_{sb}(0) \in SE(3)$, the configuration of the end-effector frame $\{b\}$ at the home configuration $\theta = 0$,
- the screw axes $\mathcal{S}_i$ in space (base) frame for each joint at $\theta = 0$, and
- the joint vector θ,

find $T_{sb}(\theta) \in SE(3)$.

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

A 3R spatial open chain



Source: Modern Robotics

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

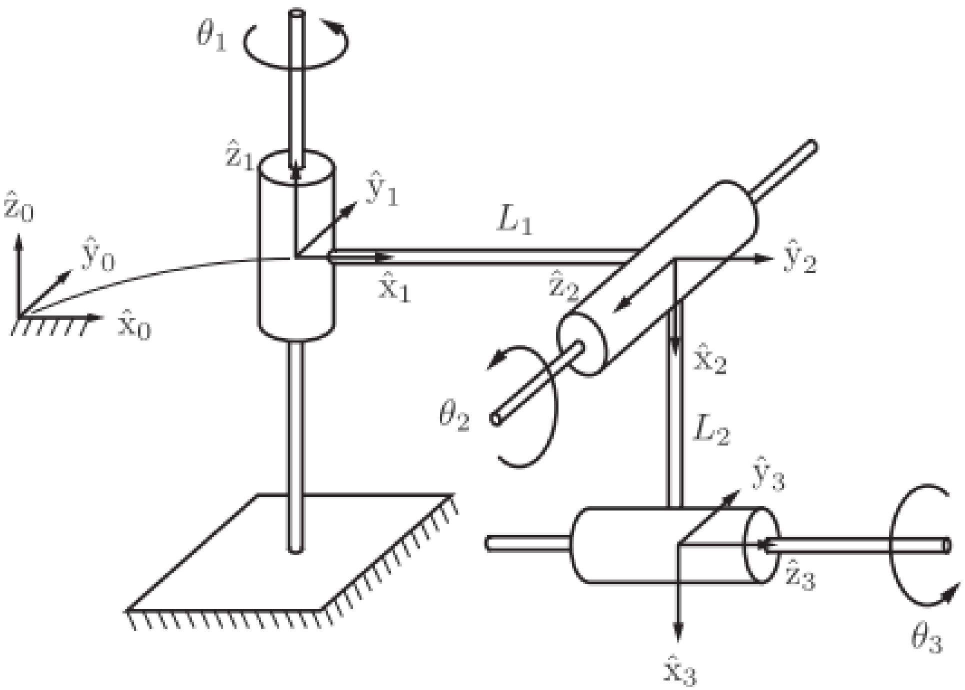
$$M = \begin{bmatrix} R_{03} & P_{03} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Note $R_{03} = [\hat{x}_{03} \quad \hat{y}_{03} \quad \hat{z}_{03}]$

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$[\mathcal{S}_1] = \begin{bmatrix} [\omega_1] & v_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

A 3R spatial open chain



Source: Modern Robotics

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

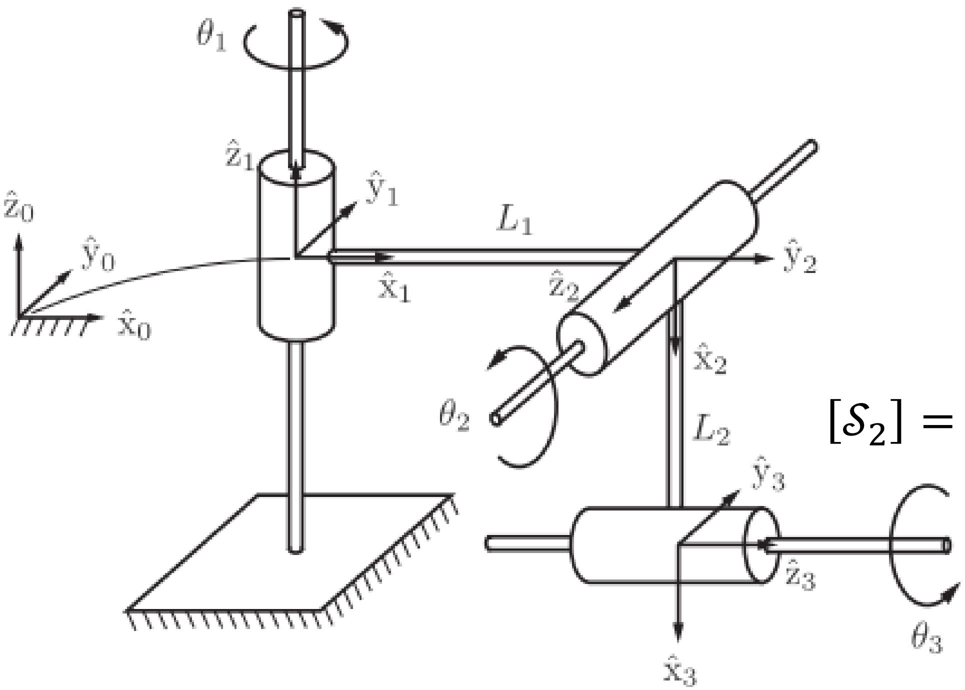
$$M = \begin{bmatrix} R_{03} & P_{03} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note $R_{03} = [\hat{x}_{03} \quad \hat{y}_{03} \quad \hat{z}_{03}]$

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\mathcal{S}_1] = \begin{bmatrix} [\omega_1] & v_1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A 3R spatial open chain



Source: Modern Robotics

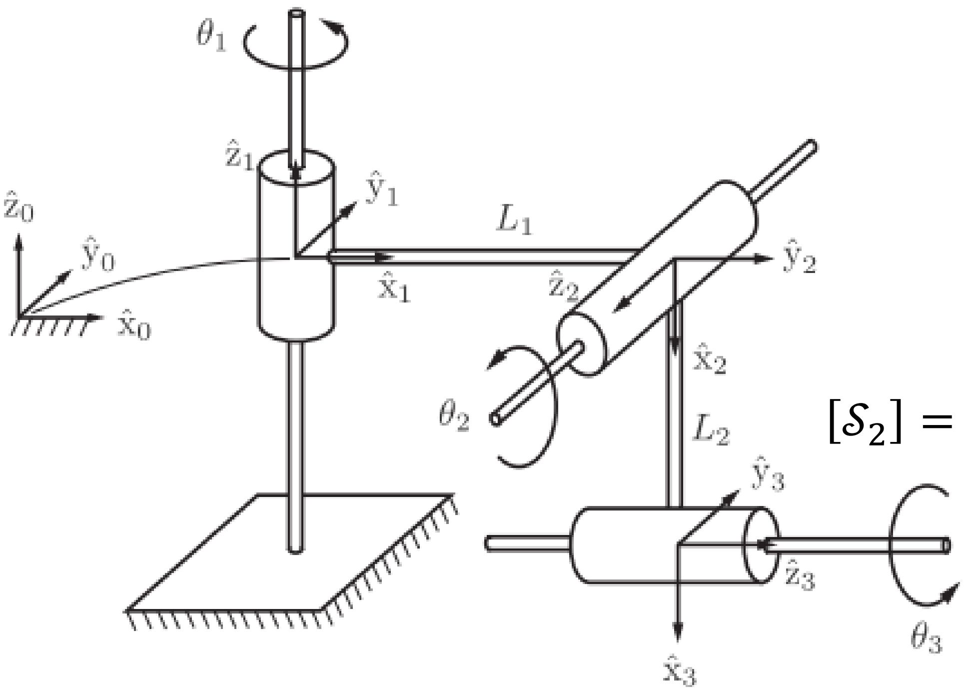
$$\mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega_3] & v_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

A 3R spatial open chain



Source: Modern Robotics

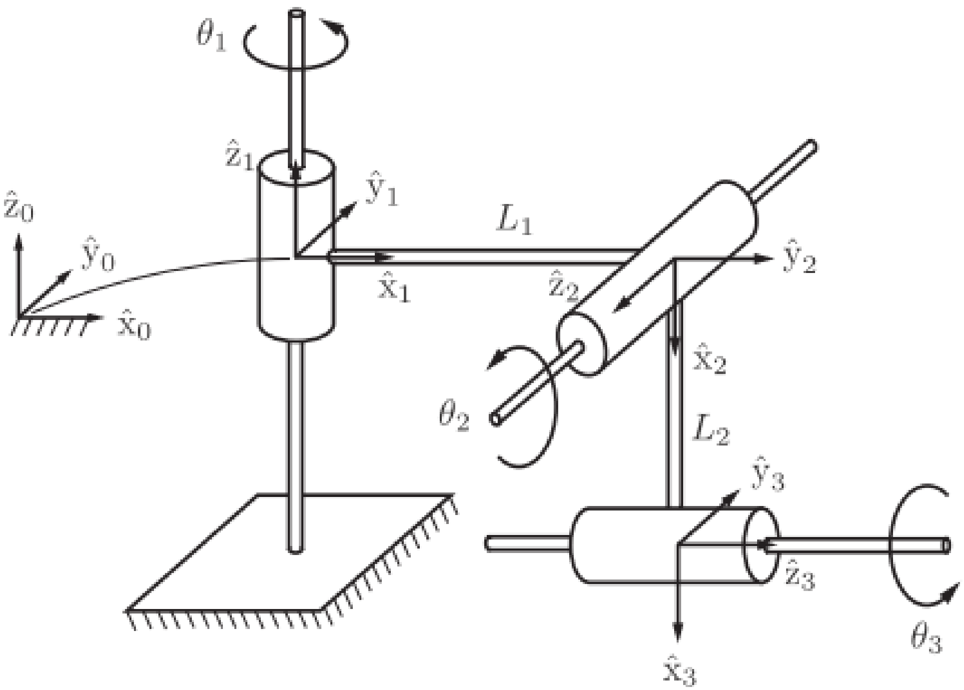
$$\mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ -L_1 \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -L_2 \\ 0 \end{bmatrix}$$

$$[\mathcal{S}_3] = \begin{bmatrix} [\omega_3] & v_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A 3R spatial open chain



Source: Modern Robotics

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,L_2,0)$

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

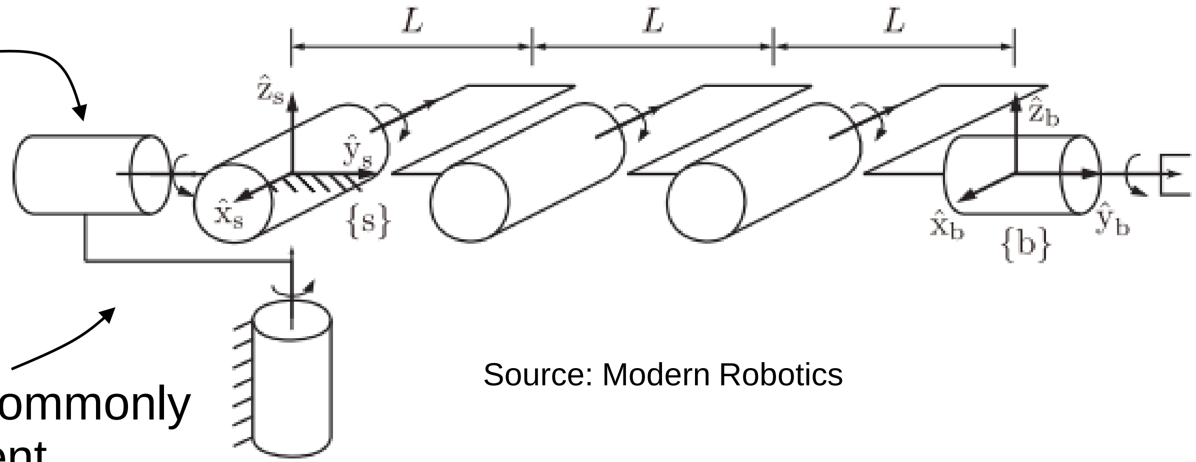
$$[\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M$$

A 6R spatial open chain

Revolute joint



Source: Modern Robotics

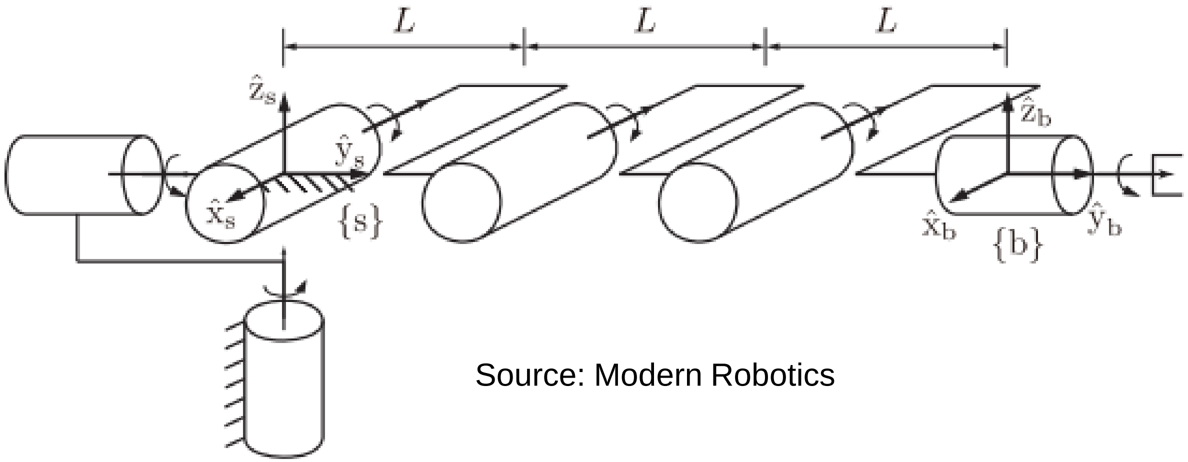
Stick figure is commonly used to represent robotic systems with articulated rigid bodies.

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

$$M = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$\mathcal{S}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{S}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{S}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

A 6R spatial open chain



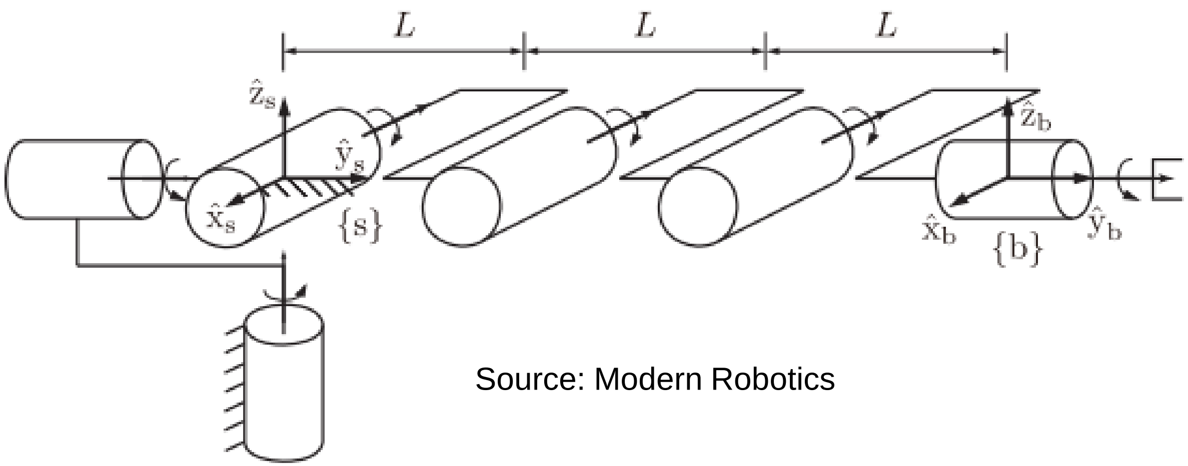
Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ L \end{bmatrix} \quad \mathcal{S}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2L \end{bmatrix} \quad \mathcal{S}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A 6R spatial open chain

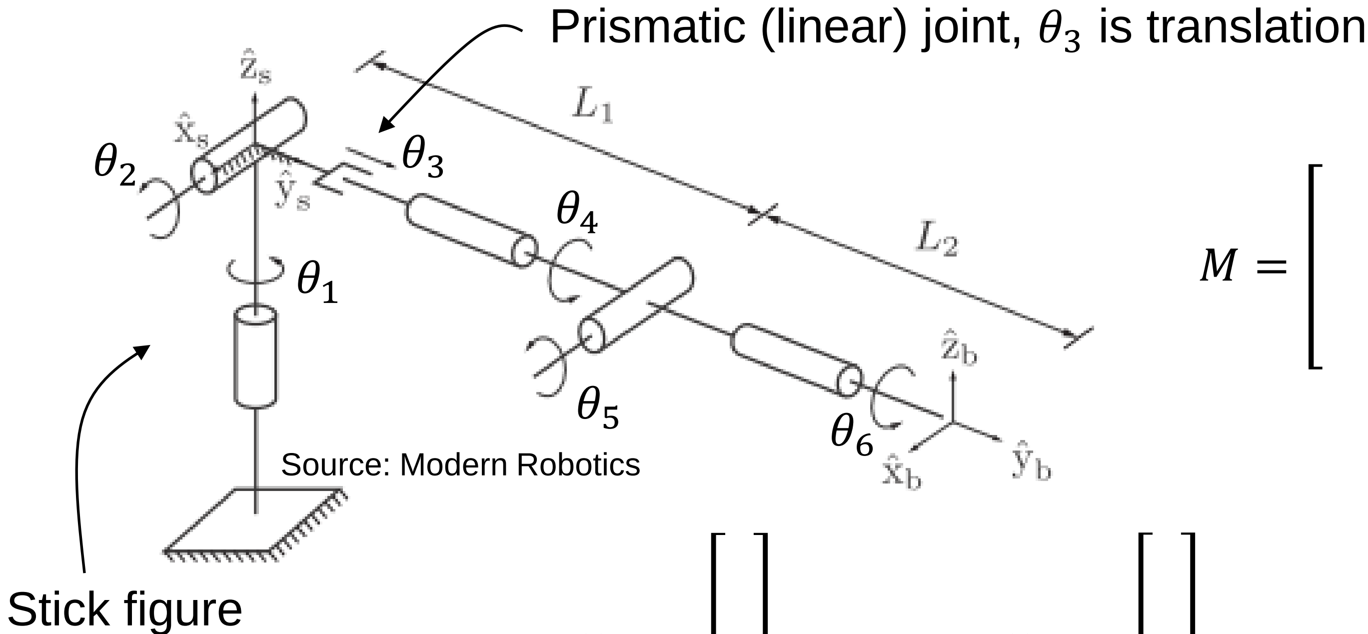


Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,0)$
4	$(-1,0,0)$	$(0,0,L)$
5	$(-1,0,0)$	$(0,0,2L)$
6	$(0,1,0)$	$(0,0,0)$

An RRPRRR spatial open chain



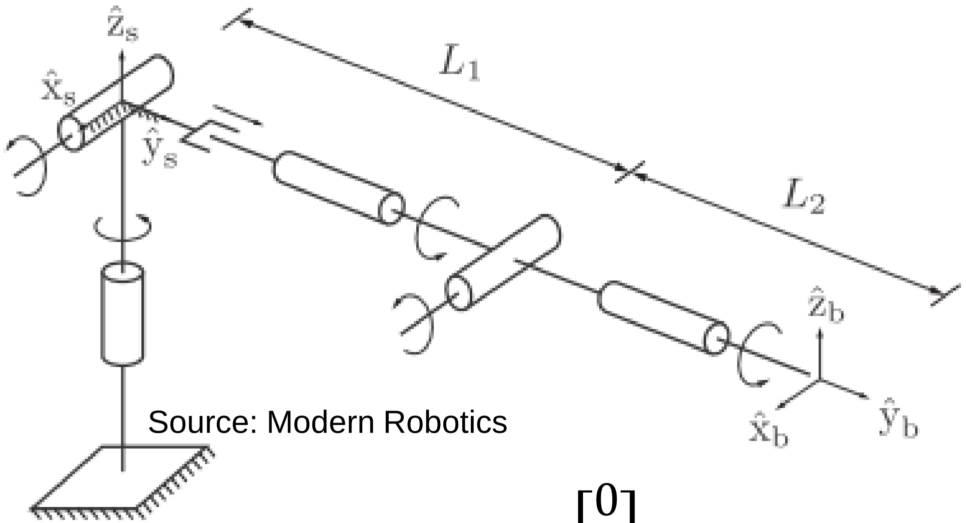
Source: Modern Robotics

$$M = \begin{bmatrix} \quad \end{bmatrix}$$

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$\mathcal{S}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{S}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{S}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

An RRPRRR spatial open chain



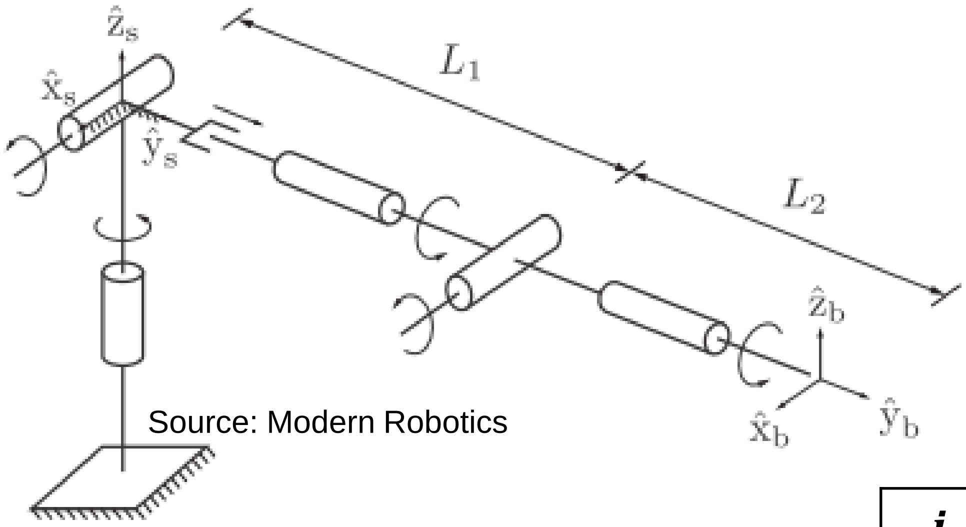
Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathcal{S}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{S}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -L_1 \end{bmatrix} \quad \mathcal{S}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

An RRPRRR spatial open chain

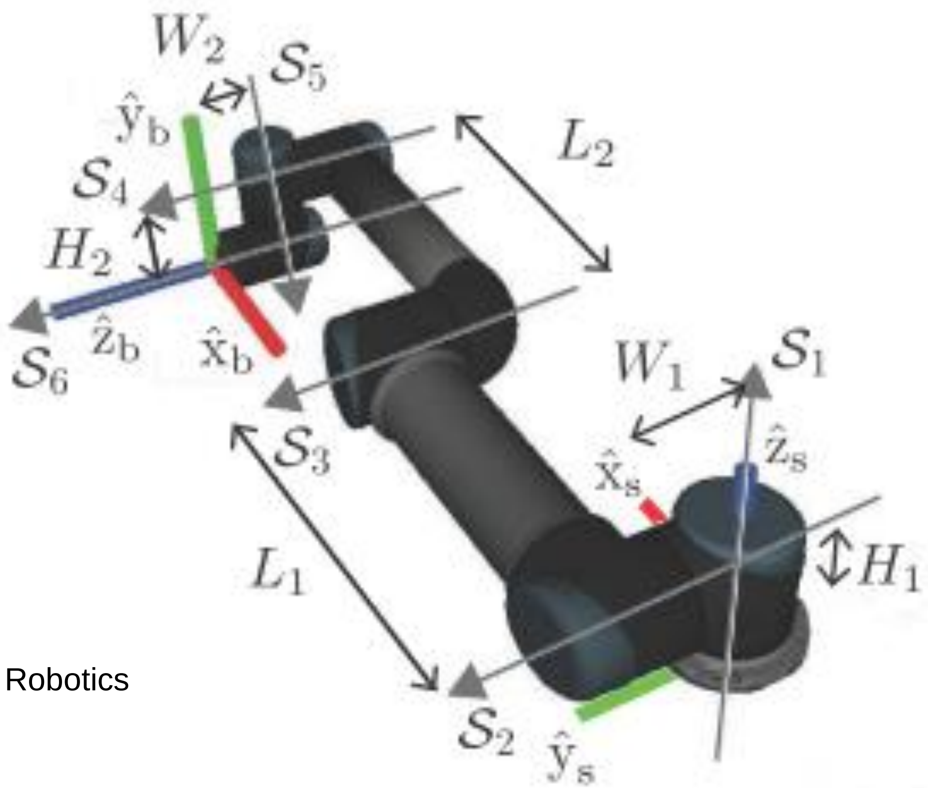


Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(1,0,0)	(0,0,0)
3	(0,0,0)	(0,1,0)
4	(0,1,0)	(0,0,0)
5	(1,0,0)	$(0,0,-L_1)$
6	(0,1,0)	(0,0,0)

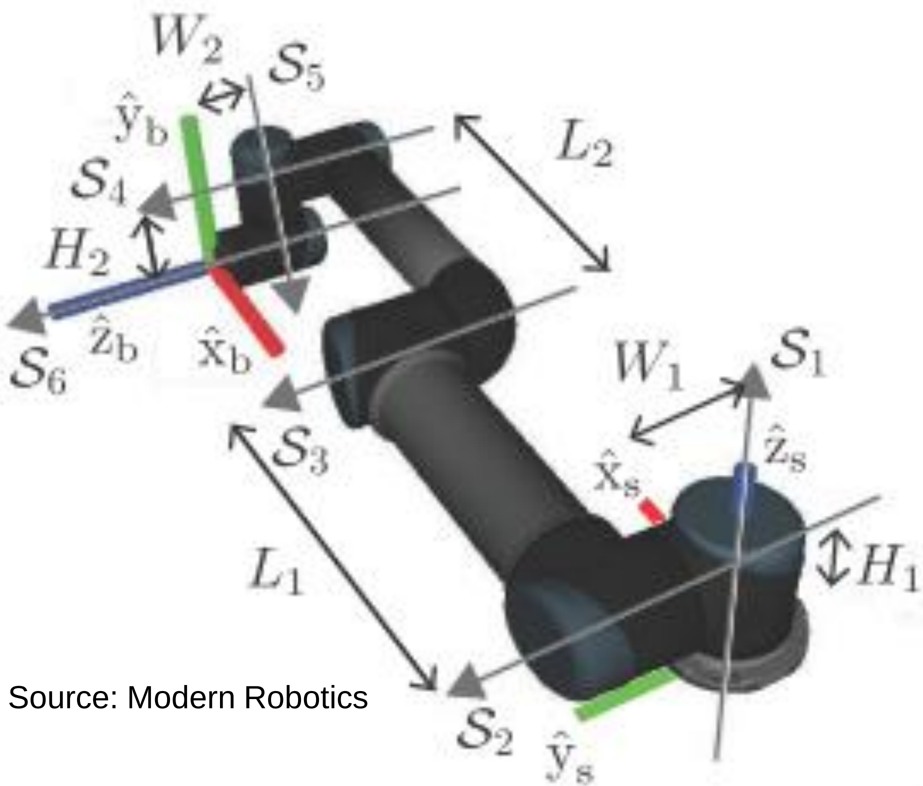
Universal Robots' UR5 6R robot arm



Source: Modern Robotics

$W_1 = 109mm$,$W_2 = 82mm$, $L_1 = 425mm$, $L_1 = 392mm$, $H_1 = 89mm$, $H_2 = 95mm$

Universal Robots' UR5 6R robot arm



Source: Modern Robotics

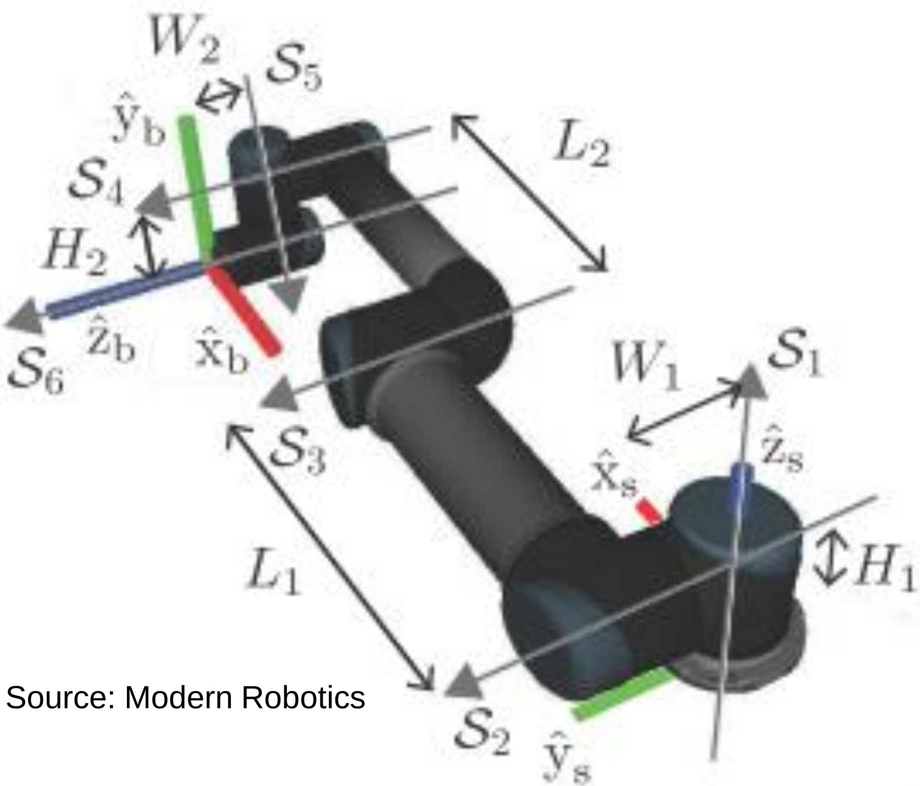
$$\mathcal{S}_1 = \begin{bmatrix}\omega_1\\ v_1\end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} \qquad \mathcal{S}_2 = \begin{bmatrix}\omega_2\\ v_2\end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

$$\mathcal{S}_3 = \begin{bmatrix}\omega_3\\ v_3\end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} \qquad \mathcal{S}_4 = \begin{bmatrix}\omega_4\\ v_4\end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

$$\mathcal{S}_5 = \begin{bmatrix}\omega_5\\ v_5\end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} \qquad \mathcal{S}_6 = \begin{bmatrix}\omega_6\\ v_6\end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

$$M = \begin{bmatrix} \\ \end{bmatrix}$$

Universal Robots' UR5 6R robot arm



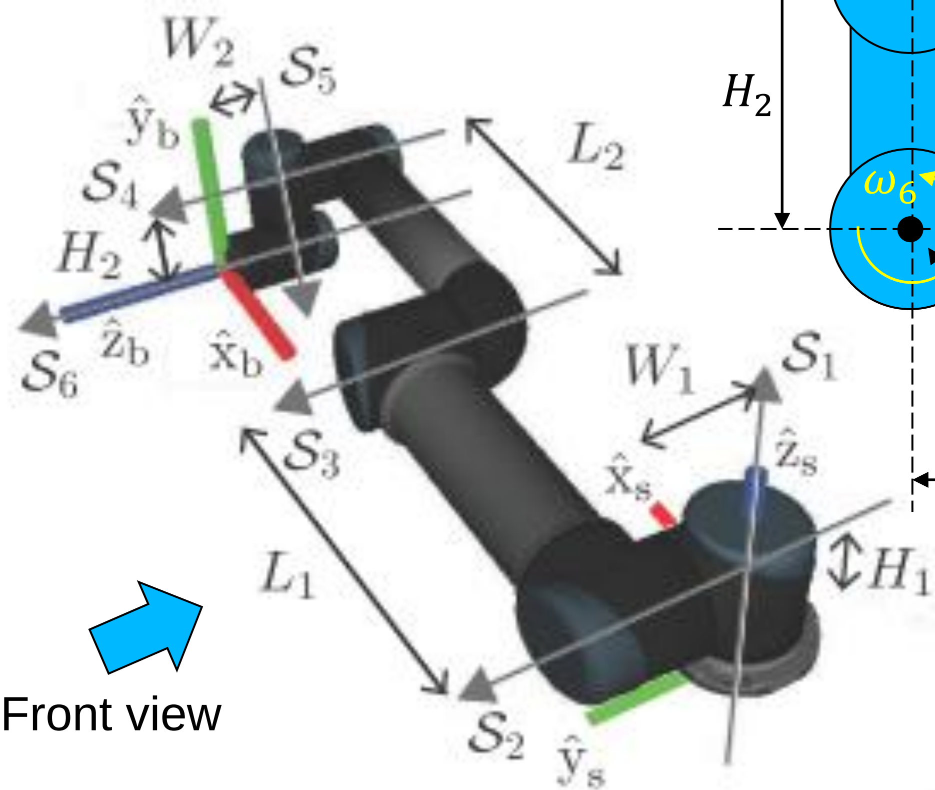
Source: Modern Robotics

$$\mathcal{S}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{S}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -H_1 \\ 0 \\ 0 \end{bmatrix}$$

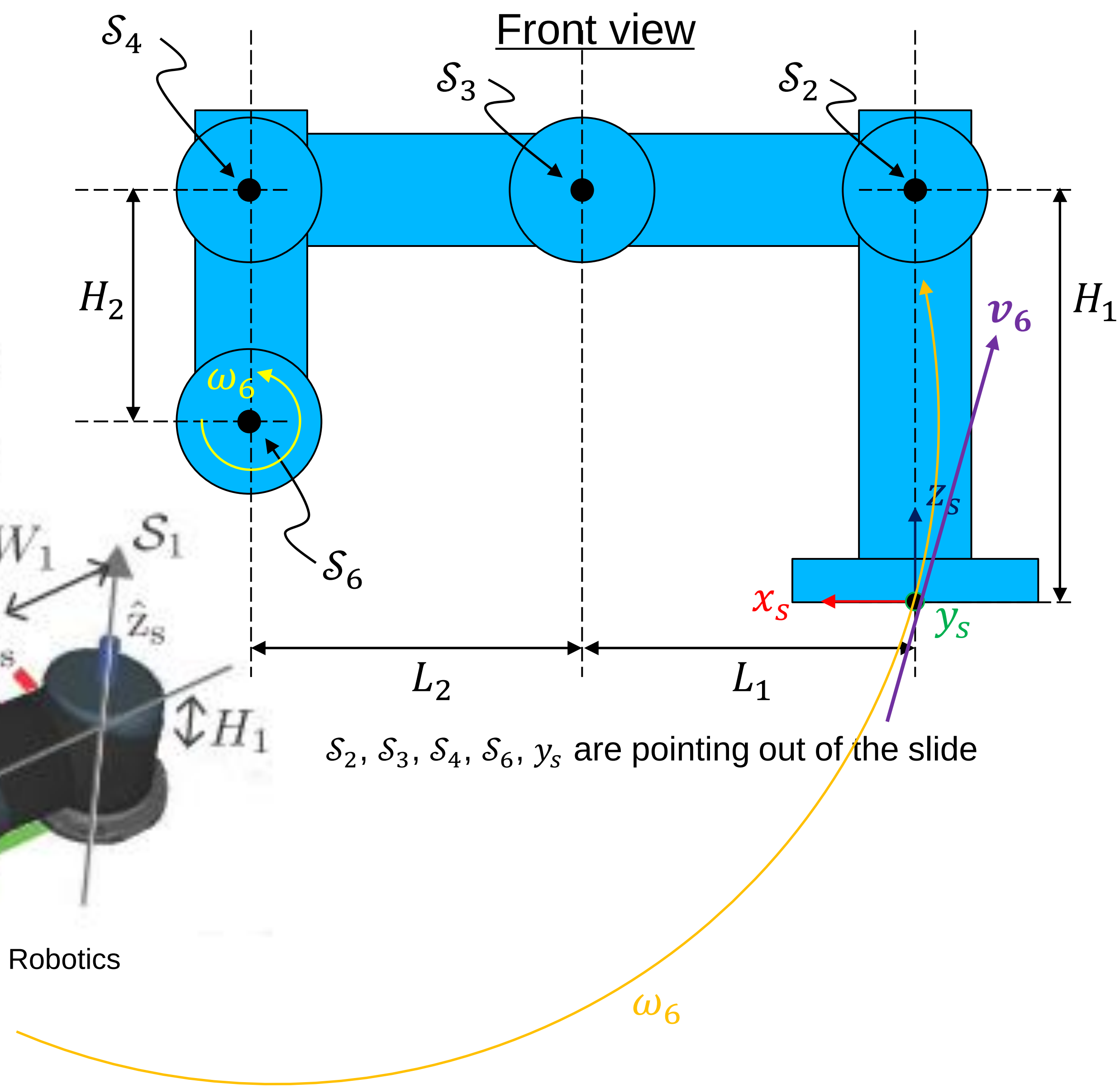
$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -H_1 \\ 0 \\ L_1 \end{bmatrix} \qquad \mathcal{S}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -H_1 \\ 0 \\ L_1 + L_2 \end{bmatrix}$$

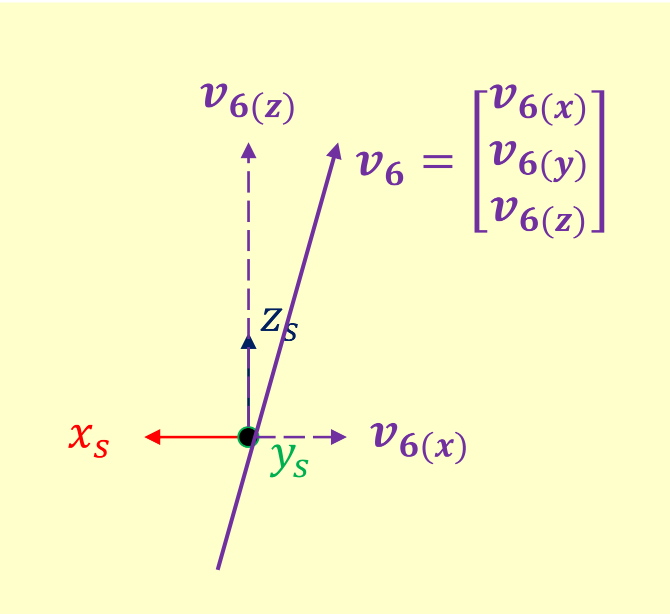
$$\mathcal{S}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -W_1 \\ L_1 + L_2 \\ 0 \end{bmatrix} \qquad \mathcal{S}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -(H_1 - H_2) \\ 0 \\ L_1 + L_2 \end{bmatrix}$$

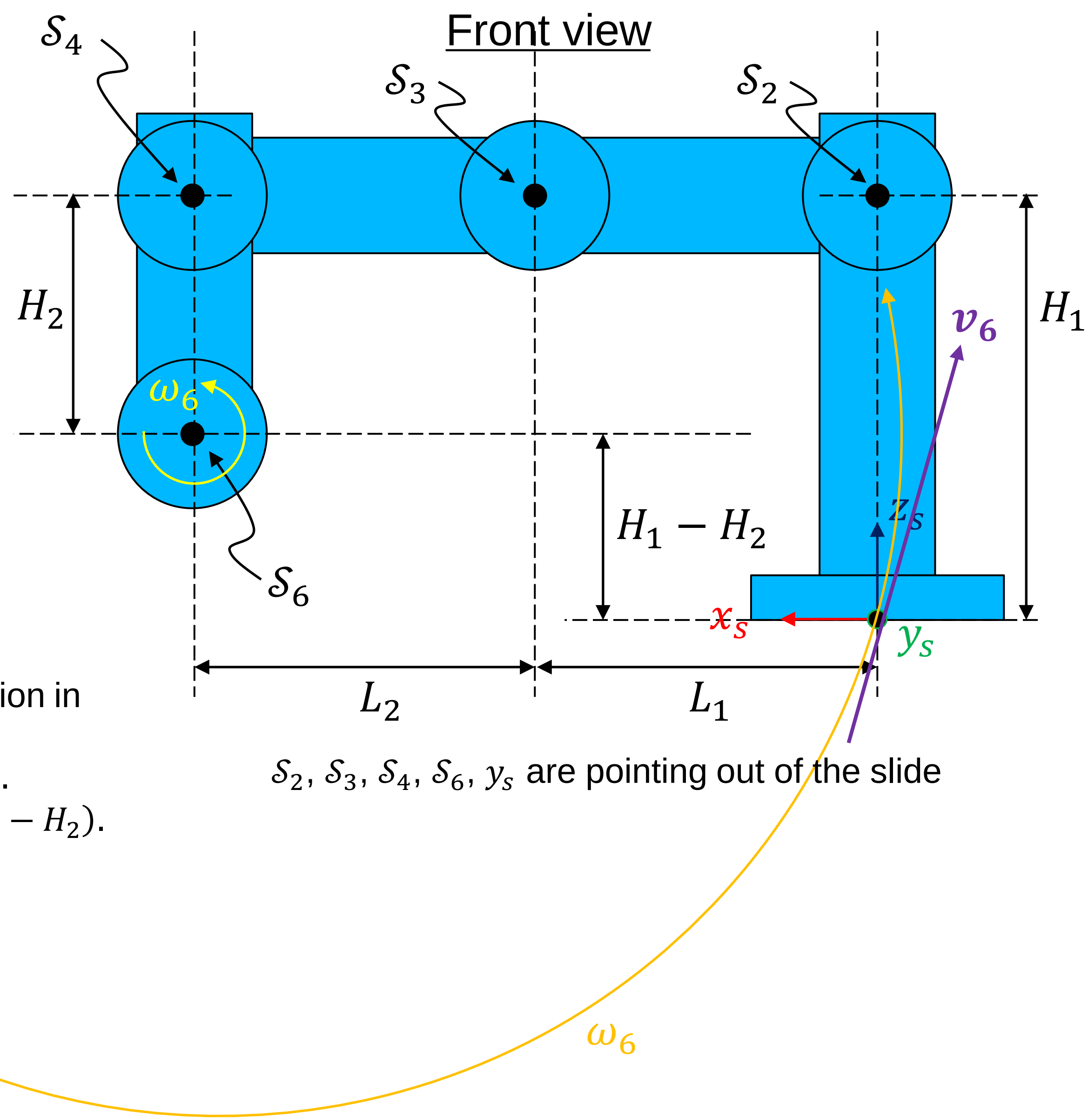
$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Source: Modern Robotics





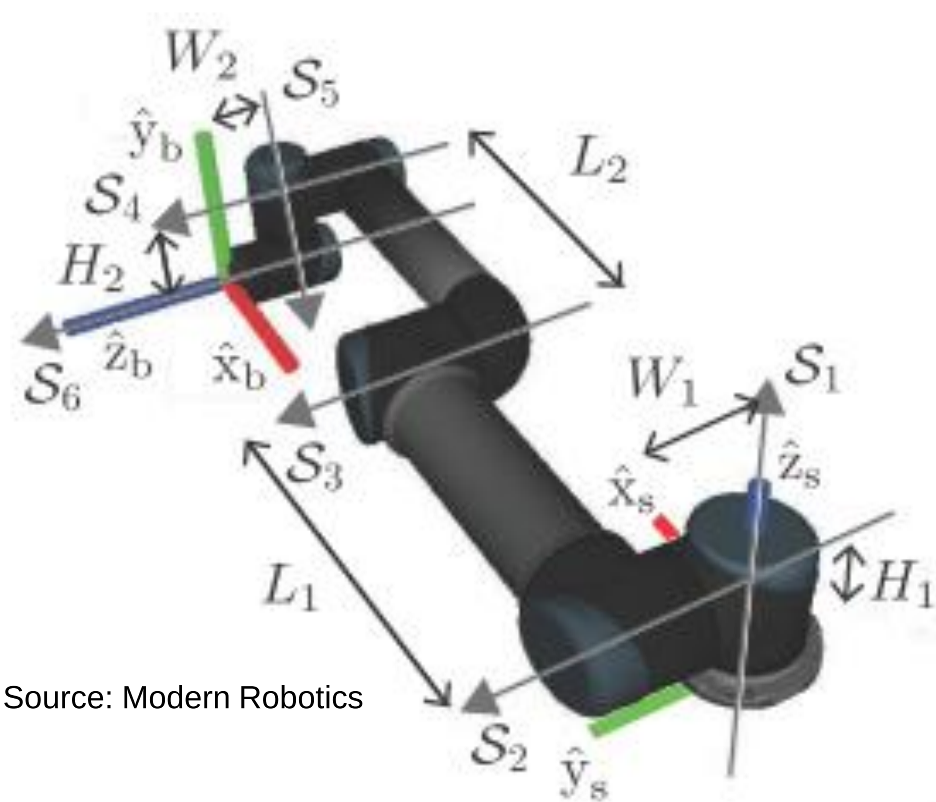


$\mathcal{S}_6$ is parallel to y_s, so there is no translation in y_s direction, i.e. $v_{6(y)} = 0$.
$v_{6(z)}$ is in z_s direction, i.e. $v_{6(z)} = L_1 + L_2$.
$v_{6(x)}$ is in $-x_s$ direction. i.e. $v_{6(x)} = -(H_1 - H_2)$.

$\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{S}_6$, y_s are pointing out of the slide

$$v_6 = \begin{bmatrix} -(H_1 - H_2) \\ 0 \\ L_1 + L_2 \end{bmatrix}$$

Universal Robots' UR5 6R robot arm



Source: Modern Robotics

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1, 0,0)$
3	$(0,1,0)$	$(-H_1, 0, L_1)$
4	$(0,1,0)$	$(-H_1, 0, L_1 + L_2)$
5	$(0,0,-1)$	$(-W_1, L_1 + L_2, 0)$
6	$(0,1,0)$	$(H_2 - H_1, 0, L_1 + L_2)$

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M$$

Universal Robots' UR5 6R robot arm

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1, 0,0)$
3	$(0,1,0)$	$(-H_1, 0, L_1)$
4	$(0,1,0)$	$(-H_1, 0, L_1 + L_2)$
5	$(0,0,-1)$	$(-W_1, L_1 + L_2, 0)$
6	$(0,1,0)$	$(H_2 - H_1, 0, L_1 + L_2)$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}\,, 0,0, {}^{\pi}/_{2}\,, 0)$

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M$$

Universal Robots' UR5 6R robot arm

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{i}$	$\boldsymbol{\omega_i}$	$\boldsymbol{v_i}$
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1, 0,0)$
3	$(0,1,0)$	$(-H_1, 0, L_1)$
4	$(0,1,0)$	$(-H_1, 0, L_1 + L_2)$
5	$(0,0,-1)$	$(-W_1, L_1 + L_2, 0)$
6	$(0,1,0)$	$(H_2 - H_1, 0, L_1 + L_2)$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}, 0,0, {}^{\pi}/_{2}, 0)$

$$\begin{aligned} T(\theta) &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M \\ &= I e^{[\mathcal{S}_2]\theta_2} I^2 e^{[\mathcal{S}_5]\theta_5} I M \\ &= e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_5]\theta_5} M \\ &= e^{-[\mathcal{S}_2]\pi/2}\, e^{[\mathcal{S}_5]\pi/2} M \end{aligned}$$

Universal Robots' UR5 6R robot arm

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1, 0,0)$
3	$(0,1,0)$	$(-H_1, 0, L_1)$
4	$(0,1,0)$	$(-H_1, 0, L_1 + L_2)$
5	$(0,0,-1)$	$(-W_1, L_1 + L_2, 0)$
6	$(0,1,0)$	$(H_2 - H_1, 0, L_1 + L_2)$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}, 0,0, {}^{\pi}/_{2}, 0)$

$W_1 = 109mm, W_2 = 82mm,\ L_1 = 425mm,\ L_2 = 392mm,\ H_1 = 89mm,\ H_2 = 95mm$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \qquad [\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} =$$

Universal Robots' UR5 6R robot arm

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1, 0,0)$
3	$(0,1,0)$	$(-H_1, 0, L_1)$
4	$(0,1,0)$	$(-H_1, 0, L_1 + L_2)$
5	$(0,0,-1)$	$(-W_1, L_1 + L_2, 0)$
6	$(0,1,0)$	$(H_2 - H_1, 0, L_1 + L_2)$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}, 0,0, {}^{\pi}/_{2}, 0)$

$W_1 = 109mm, W_2 = 82mm,\ L_1 = 425mm,\ L_2 = 392mm,\ H_1 = 89mm,\ H_2 = 95mm$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

$$[\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$[\mathcal{S}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -H_1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -0.089 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Universal Robots' UR5 6R robot arm

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & 1 & -0.089 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}, 0, 0, {}^{\pi}/_{2}, 0)$

$W_1 = 109mm, W_2 = 82mm,\ L_1 = 425mm,\ L_2 = 392mm,\ H_1 = 89mm,\ H_2 = 95mm$

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

$$e^{-[\mathcal{S}_2]^{\pi}/_{2}} = \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \sin \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \\ \\ \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix}\left(1 - \cos \quad \right) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Universal Robots' UR5 6R robot arm

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & 1 & -0.089 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}, 0, 0, {}^{\pi}/_{2}, 0)$

$W_1 = 109mm, W_2 = 82mm,\ L_1 = 425mm,\ L_2 = 392mm,\ H_1 = 89mm,\ H_2 = 95mm$

$$e^{[\widehat{\omega}]\theta} = I + \sin\theta[\widehat{\omega}] + (1 - \cos\theta)[\widehat{\omega}]^2$$

$$e^{-[\mathcal{S}_2]\pi/2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sin\frac{-\pi}{2} \begin{bmatrix} 0 & 0 & 1 & -0.089 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\left[\begin{bmatrix} 0 & 0 & 1 & -0.089 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right] \begin{bmatrix} 0 & 0 & 1 & -0.089 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(1 - \cos\frac{-\pi}{2}\right) = \begin{bmatrix} 0 & 0 & -1 & 0.089 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.089 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Universal Robots' UR5 6R robot arm

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1, 0,0)$
3	$(0,1,0)$	$(-H_1, 0, L_1)$
4	$(0,1,0)$	$(-H_1, 0, L_1 + L_2)$
5	$(0,0,-1)$	$(-W_1, L_1 + L_2, 0)$
6	$(0,1,0)$	$(H_2 - H_1, 0, L_1 + L_2)$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}, 0, 0, {}^{\pi}/_{2}, 0)$

$W_1 = 109mm$, $W_2 = 82mm$, $L_1 = 425mm$, $L_2 = 392mm$, $H_1 = 89mm$, $H_2 = 95mm$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \qquad [\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$[\mathcal{S}_5] = \begin{bmatrix} [\omega_5] & v_5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -W_1 \\ -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -0.109 \\ -1 & 0 & 0 & 0.817 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Universal Robots' UR5 6R robot arm

$$[\mathcal{S}_5] = \begin{bmatrix} 0 & 1 & 0 & -0.109 \\ -1 & 0 & 0 & 0.817 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -{}^{\pi}/_{2}, 0, 0, {}^{\pi}/_{2}, 0)$

$W_1 = 109mm, W_2 = 82mm,\ L_1 = 425mm,\ L_2 = 392mm,\ H_1 = 89mm,\ H_2 = 95mm$

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

$$e^{[\mathcal{S}_5]\pi/2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sin\frac{\pi}{2}\begin{bmatrix} 0 & 1 & 0 & -0.109 \\ -1 & 0 & 0 & 0.817 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 1 & 0 & -0.109 \\ -1 & 0 & 0 & 0.817 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & -0.109 \\ -1 & 0 & 0 & 0.817 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\left(1 - \cos\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 1 & 0 & 0.708 \\ -1 & 0 & 0 & 0.926 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Universal Robots' UR5 6R robot arm

$$M = \begin{bmatrix} -1 & 0 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & W_1 + W_2 \\ 0 & 1 & 0 & H_1 - H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,1,0)$	$(-H_1, 0,0)$
3	$(0,1,0)$	$(-H_1, 0, L_1)$
4	$(0,1,0)$	$(-H_1, 0, L_1 + L_2)$
5	$(0,0,-1)$	$(-W_1, L_1 + L_2, 0)$
6	$(0,1,0)$	$(H_2 - H_1, 0, L_1 + L_2)$

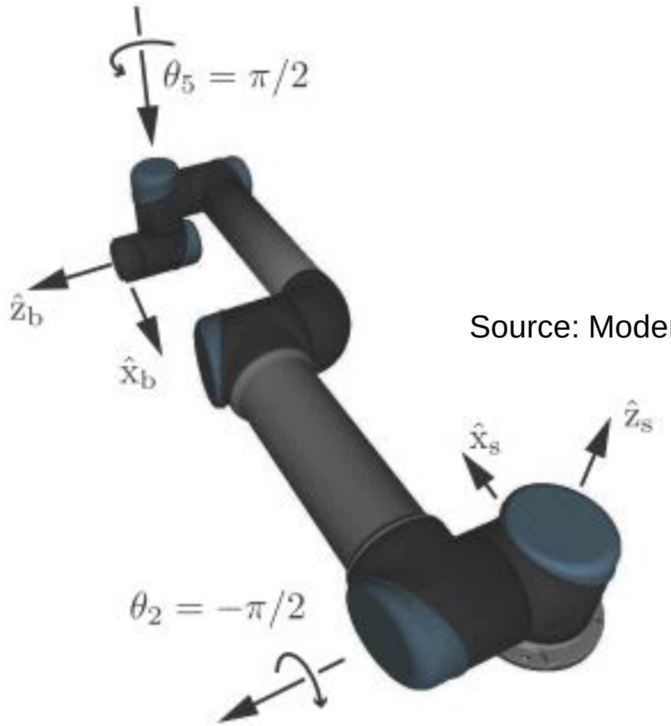
Determine $T(\theta)$, i.e. pose of the end-effector given $\theta = (0, -\pi/2, 0, 0, \pi/2, 0)$

$$e^{[\mathcal{S}_2]\pi/2} = \begin{bmatrix} 0 & 0 & -1 & 0.089 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.089 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad e^{[\mathcal{S}_5]\pi/2} = \begin{bmatrix} 0 & 1 & 0 & 0.708 \\ -1 & 0 & 0 & 0.926 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

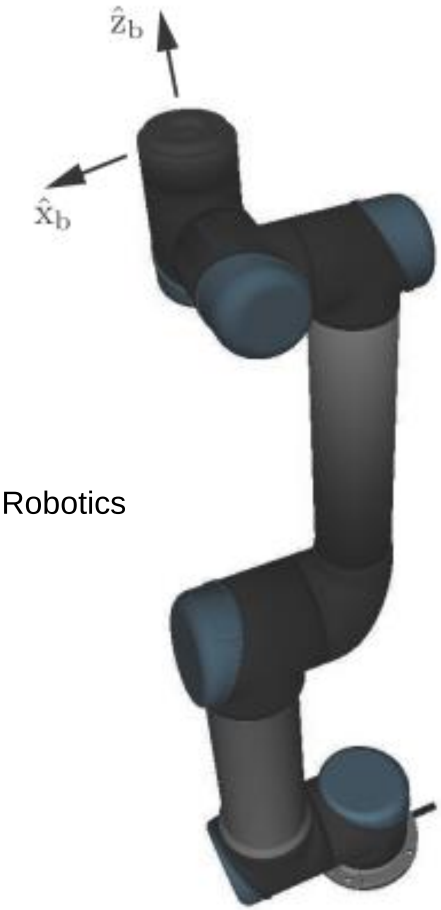
where the linear units are meters, and

$$T(\theta) = e^{-[\mathcal{S}_2]\pi/2} \, e^{[\mathcal{S}_5]\pi/2} M = \begin{bmatrix} 0 & -1 & 0 & 0.095 \\ 1 & 0 & 0 & 0.109 \\ 0 & 0 & 1 & 0.988 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Universal Robots' UR5 6R robot arm



Source: Modern Robotics



M

$$T(\theta) = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_5]\theta_5} M$$
$$\theta = \left(0, -{}^{\pi}/_{2}, 0, 0, {}^{\pi}/_{2}, 0\right)$$

Power of Exponentials (PoE): body form

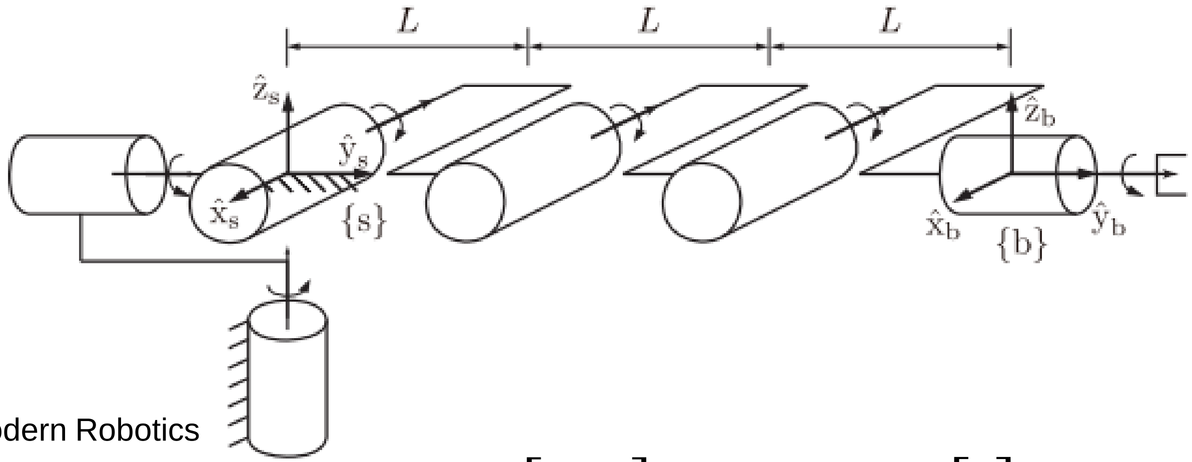
Given

- $M = T_{sb}(0) \in SE(3)$, the configuration of the end-effector frame $\{\mathrm{b}\}$ at the home configuration $\theta = 0$,
- the screw axes $\mathcal{B}_i$ in body (end-effector) frame for each joint at $\theta = 0$, and
- the joint vector θ,

find $T_{sb}(\theta) \in SE(3)$.

$$T(\theta) = Me^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n}$$

A 6R spatial open chain: body form



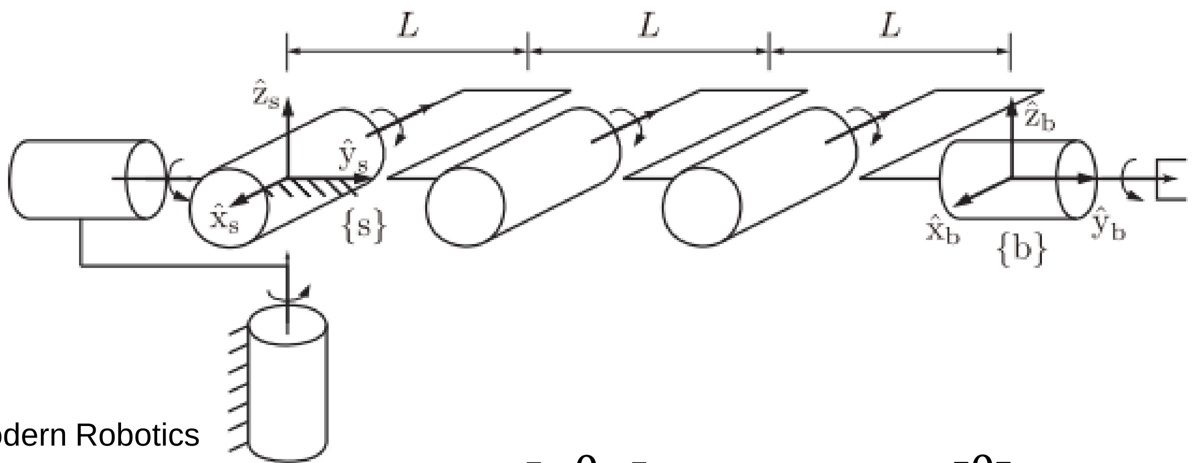
Source: Modern Robotics

$$\mathcal{B}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{B}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{B}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$M = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

$$\mathcal{B}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{B}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} \quad \mathcal{B}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

A 6R spatial open chain: body form



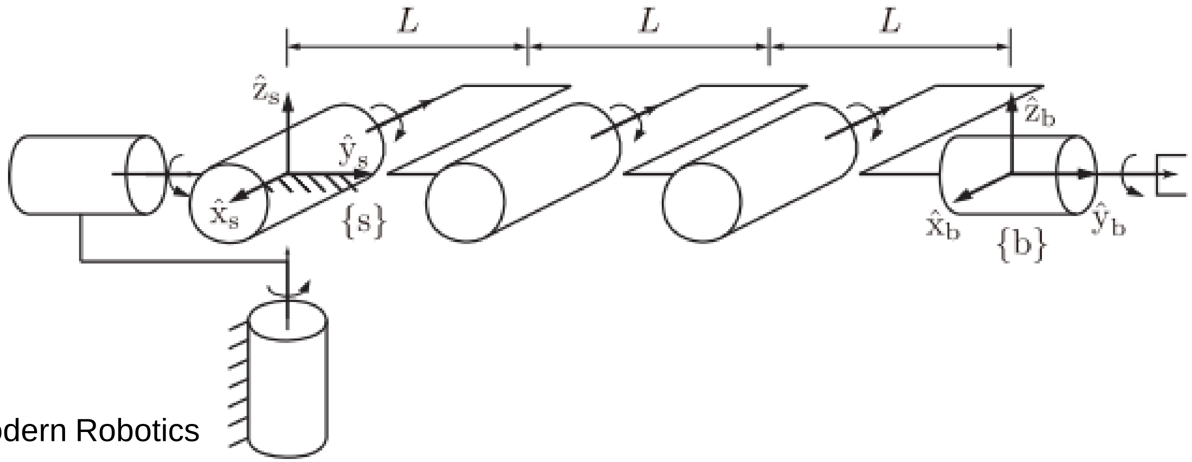
Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3L \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{B}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathcal{B}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3L \end{bmatrix}$$

$$\mathcal{B}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2L \end{bmatrix} \quad \mathcal{B}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -L \end{bmatrix} \quad \mathcal{B}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A 6R spatial open chain: body form

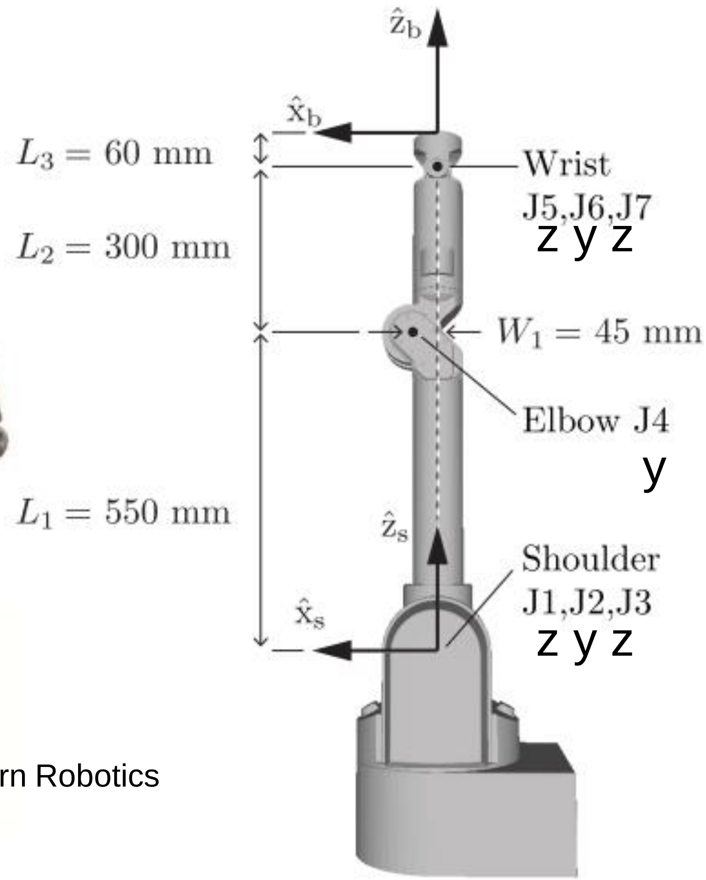


Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	$(0,0,1)$	$(-3L,0,0)$
2	$(0,1,0)$	$(0,0,0)$
3	$(-1,0,0)$	$(0,0,-3L)$
4	$(-1,0,0)$	$(0,0,-2L)$
5	$(-1,0,0)$	$(0,0,-L)$
6	$(0,1,0)$	$(0,0,0)$

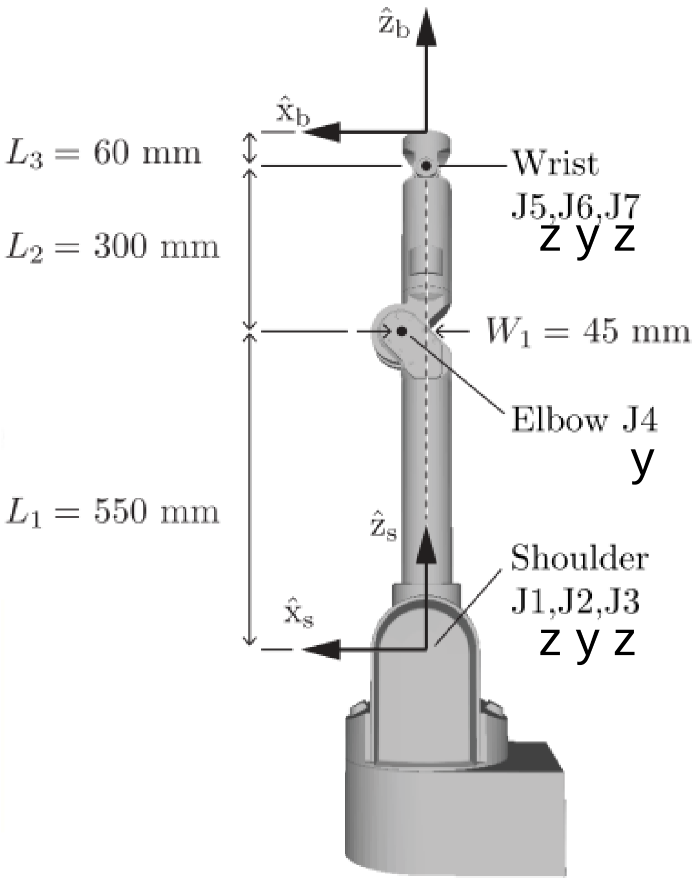
Barrett Technology's WAM 7R robot arm



Source: Modern Robotics

At the zero configuration, axes 1, 3, 5, and 7 are along z_s and axes 2, 4, and 6 are aligned with y_s out of the page. Positive rotations are given by the right-hand rule. Axes 1, 2, and 3 intersect at the origin of {s} and axes 5, 6, and 7 intersect at a point 60mm from {b}.

Barrett Technology's WAM 7R robot arm



Source: Modern Robotics

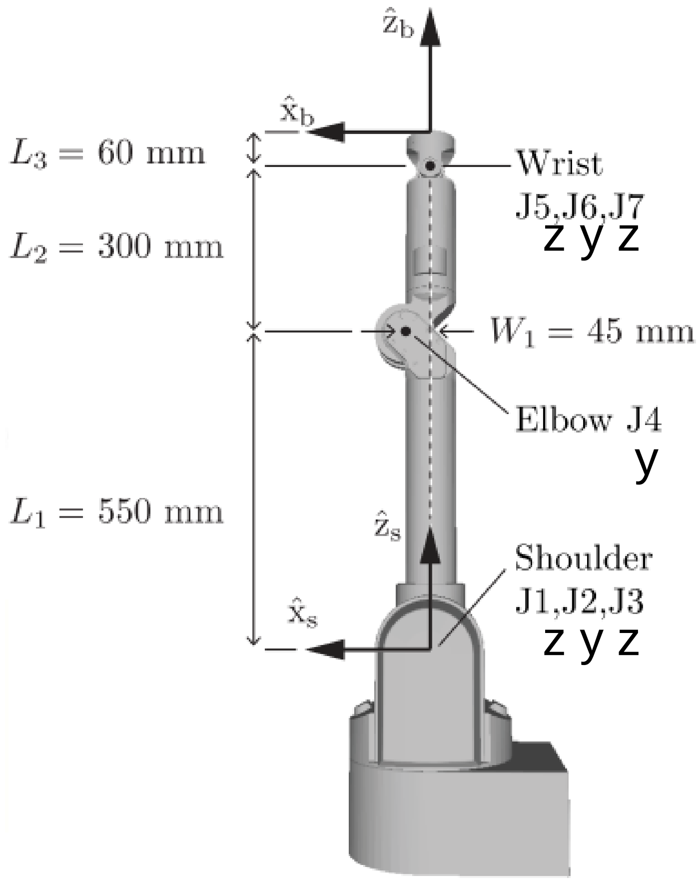
$$M = \begin{bmatrix} \quad \end{bmatrix}$$

$$\mathcal{B}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{B}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$\mathcal{B}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{B}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$\mathcal{B}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{B}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \qquad \mathcal{B}_7 = \begin{bmatrix} \omega_7 \\ v_7 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Barrett Technology's WAM 7R robot arm



Source: Modern Robotics

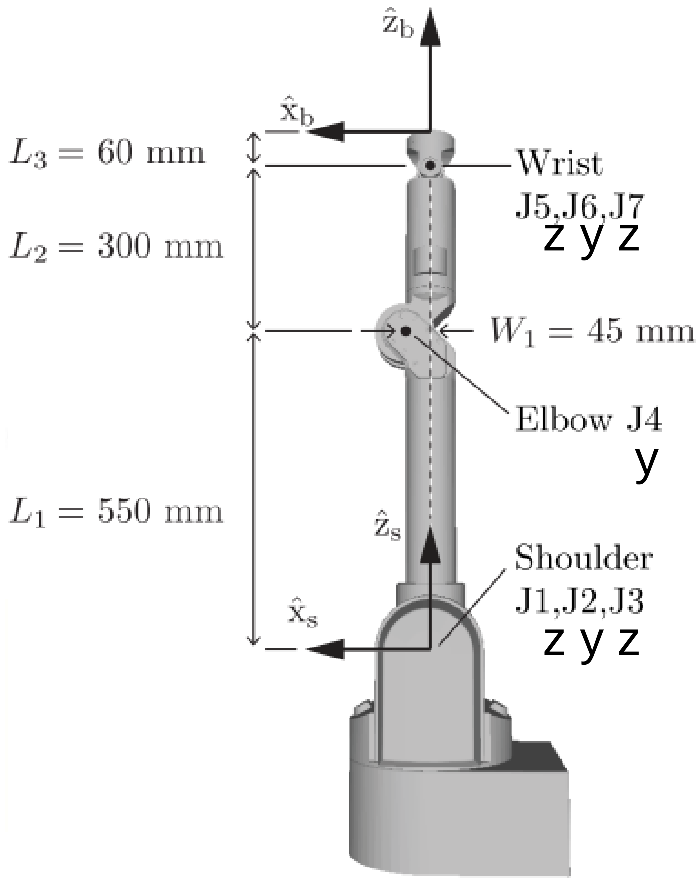
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{B}_1 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{B}_2 = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ L_1 + L_2 + L_3 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{B}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{B}_4 = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ L_2 + L_3 \\ 0 \\ W_1 \end{bmatrix}$$

$$\mathcal{B}_5 = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{B}_6 = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ L_3 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{B}_7 = \begin{bmatrix} \omega_7 \\ v_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Barrett Technology's WAM 7R robot arm



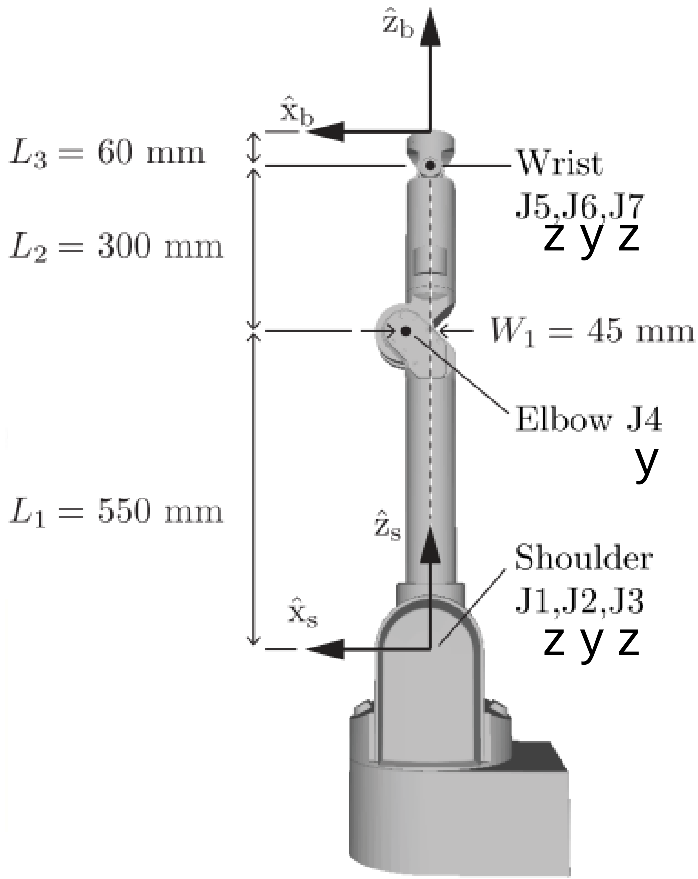
Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,1,0)	$(L_1 + L_2 + L_3, 0,0)$
3	(0,0,1)	(0,0,0)
4	(0,1,0)	$(L_2 + L_3, 0, W_1)$
5	(0,0,1)	(0,0,0)
6	(0,1,0)	$(L_3, 0,0)$
7	(0,0,1)	(0,0,0)

$$L_1 = 550mm, L_2 = 300mm, L_3 = 60mm, W_1 = 45mm$$

Barrett Technology's WAM 7R robot arm



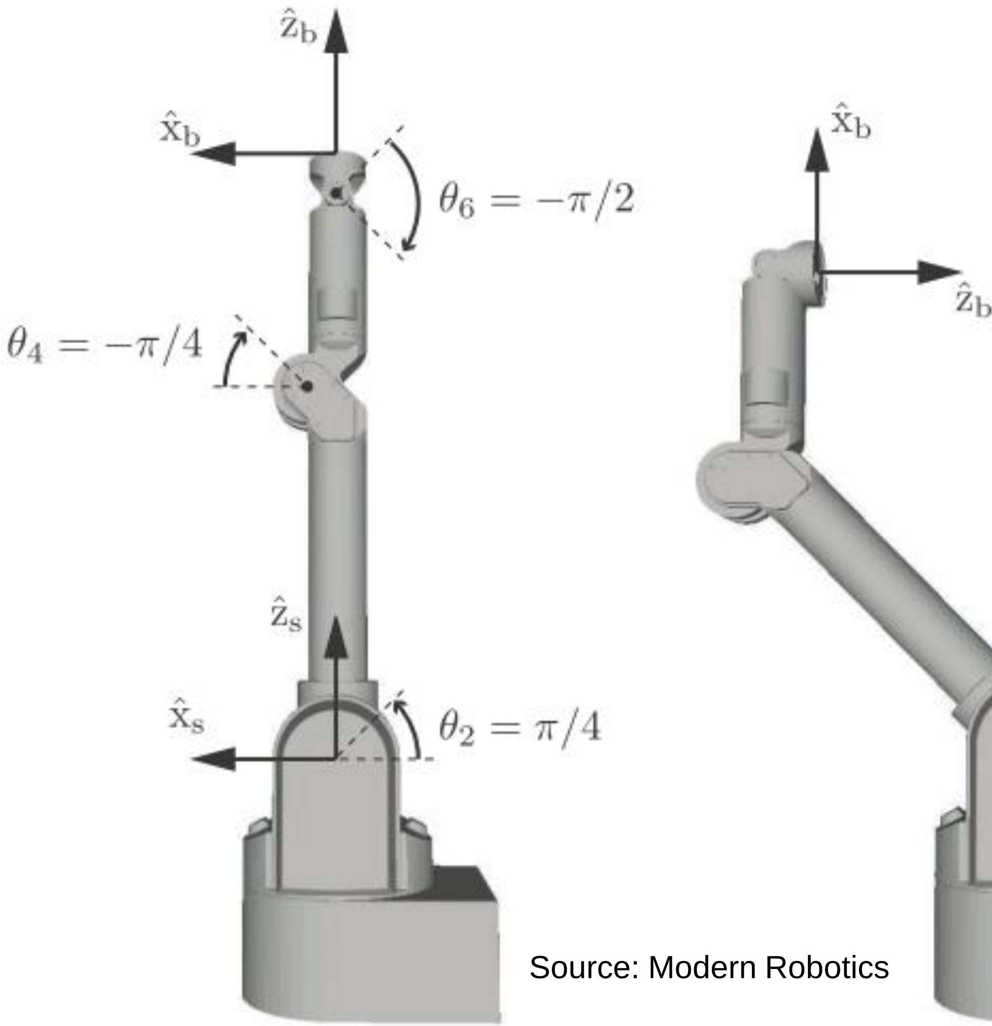
Source: Modern Robotics

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,1,0)	(0.91,0,0)
3	(0,0,1)	(0,0,0)
4	(0,1,0)	(0.36,0,0.045)
5	(0,0,1)	(0,0,0)
6	(0,1,0)	(0.06,0,0)
7	(0,0,1)	(0,0,0)

$$L_1 = 550mm, L_2 = 300mm, L_3 = 60mm, \mathrm{W}_1 = 45mm$$

Barrett Technology's WAM 7R robot arm

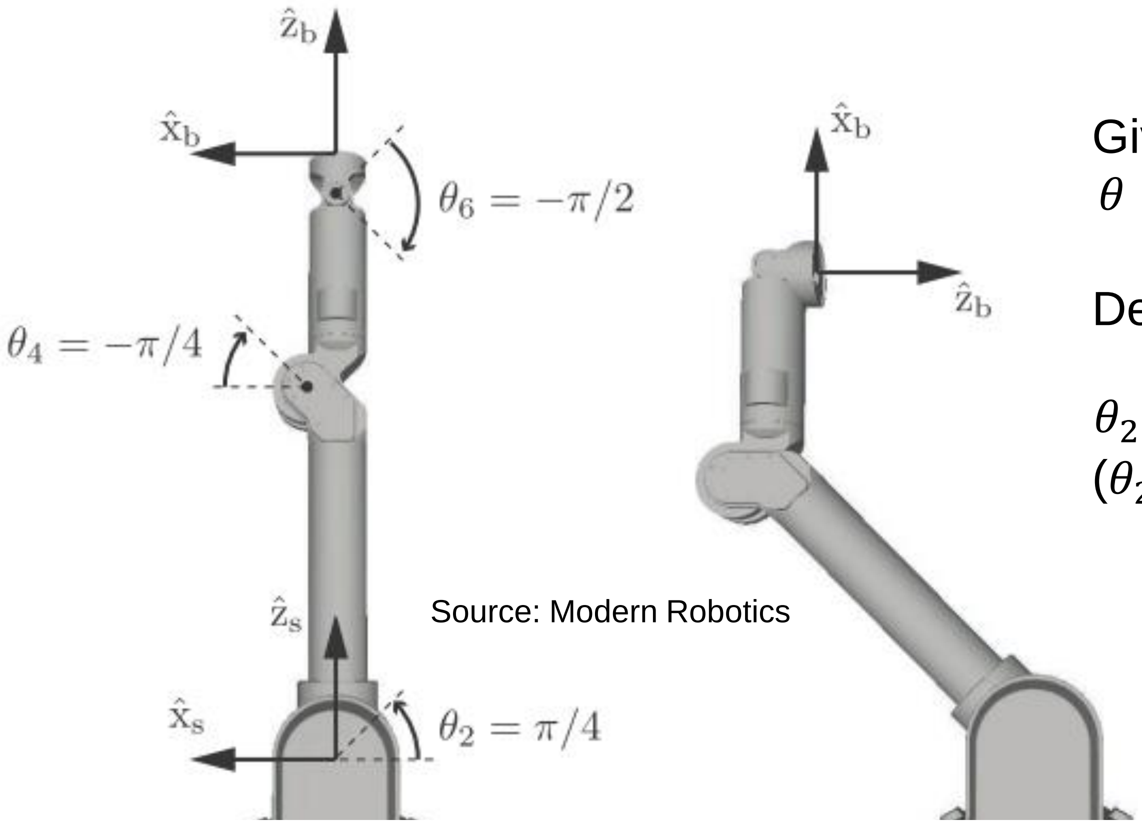


Source: Modern Robotics

Given
$\theta = \left(0, {}^{\pi}/_{4}, 0, -{}^{\pi}/_{4}, 0, -{}^{\pi}/_{2}, 0\right)$

Determine $T_{sb}(\theta)$.

Barrett Technology's WAM 7R robot arm



Source: Modern Robotics

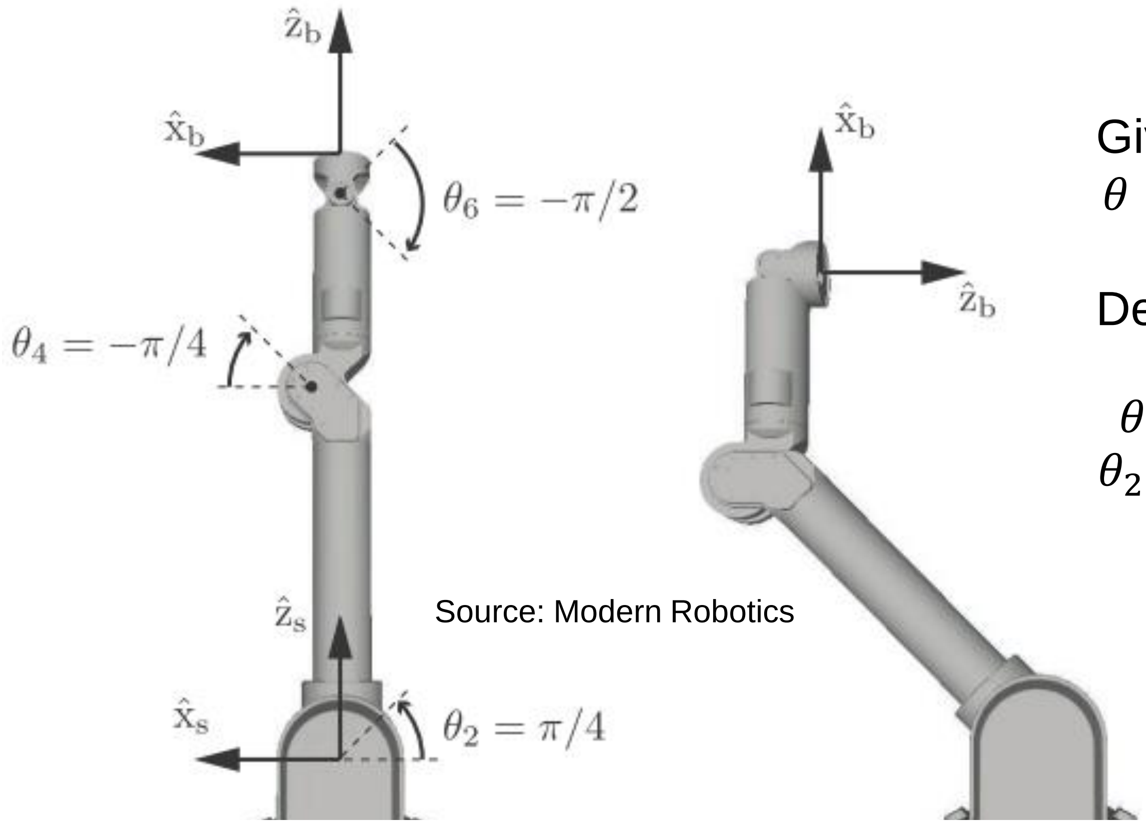
Given
$\theta = (0, \pi/4, 0, -\pi/4, 0, -\pi/2, 0)$

Determine $T_{sb}(\theta)$.

$\theta_2 = \pi/4, \theta_4 = -\pi/4, \theta_6 = -\pi/2$
$(\theta_2 = 45°, \theta_4 = -45°, \theta_6 = -90°)$

$T_{sb}(\theta)$
$= Me^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3} e^{[\mathcal{B}_4]\theta_4} e^{[\mathcal{B}_5]\theta_5} e^{[\mathcal{B}_6]\theta_6} e^{[\mathcal{B}_7]\theta_7}$
$=$

Barrett Technology's WAM 7R robot arm



Source: Modern Robotics

Given
$\theta = \left(0, {}^{\pi}/_{4}, 0, -{}^{\pi}/_{4}, 0, -{}^{\pi}/_{2}, 0\right)$

Determine $T_{sb}(\theta)$.

$\theta_2 = 45°, \theta_4 = -45°, \theta_6 = -90°$
$\theta_2 = {}^{\pi}/_{4}, \theta_4 = -{}^{\pi}/_{4}, \theta_6 = -{}^{\pi}/_{2}$

$$T_{sb}(\theta) = Me^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3} e^{[\mathcal{B}_4]\theta_4} e^{[\mathcal{B}_5]\theta_5} e^{[\mathcal{B}_6]\theta_6} e^{[\mathcal{B}_7]\theta_7} = Me^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_4]\theta_4} e^{[\mathcal{B}_6]\theta_6}$$

Barrett Technology's WAM 7R robot arm

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_2 = {}^{\pi}/_{4}, \theta_4 = -{}^{\pi}/_{4}, \theta_6 = -{}^{\pi}/_{2}$$

$$T_{sb}(\theta) = Me^{[\mathcal{B}_2]\theta_2}e^{[\mathcal{B}_4]\theta_4}e^{[\mathcal{B}_6]\theta_6}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,1,0)	(0.91,0,0)
3	(0,0,1)	(0,0,0)
4	(0,1,0)	(0.36,0,0.045)
5	(0,0,1)	(0,0,0)
6	(0,1,0)	(0.06,0,0)
7	(0,0,1)	(0,0,0)

$$[\mathcal{B}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$[\mathcal{B}_4] = \begin{bmatrix} \\ \\ \\ \end{bmatrix} \quad [\mathcal{B}_6] = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Barrett Technology's WAM 7R robot arm

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta_2 = {}^{\pi}/_{4}, \theta_4 = -{}^{\pi}/_{4}, \theta_6 = -{}^{\pi}/_{2}$$

$$T_{sb}(\theta) = Me^{[\mathcal{B}_2]\theta_2}e^{[\mathcal{B}_4]\theta_4}e^{[\mathcal{B}_6]\theta_6}$$

i	ω_i	v_i
1	(0,0,1)	(0,0,0)
2	(0,1,0)	(0.91,0,0)
3	(0,0,1)	(0,0,0)
4	(0,1,0)	(0.36,0,0.045)
5	(0,0,1)	(0,0,0)
6	(0,1,0)	(0.06,0,0)
7	(0,0,1)	(0,0,0)

$$[\mathcal{B}_2] = \begin{bmatrix} [\omega_2] & v_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathcal{B}_4] = \begin{bmatrix} 0 & 0 & 1 & 0.36 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0.045 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\mathcal{B}_6] = \begin{bmatrix} 0 & 0 & 1 & 0.06 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Barrett Technology's WAM 7R robot arm

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\mathcal{B}_2] = \begin{bmatrix} 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\theta_2 = {}^{\pi}/_{4}, \theta_4 = -{}^{\pi}/_{4}, \theta_6 = -{}^{\pi}/_{2}$$

$$T_{sb}(\theta) = M e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_4]\theta_4} e^{[\mathcal{B}_6]\theta_6}$$

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2$$

$$e^{[\mathcal{B}_2]\pi/4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sin\frac{\pi}{4}\begin{bmatrix} 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\left(1-\cos\frac{\pi}{4}\right) = \begin{bmatrix} 0.707 & 0 & 0.707 & 0.6435 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & -0.267 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Barrett Technology's WAM 7R robot arm

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\mathcal{B}_4] = \begin{bmatrix} 0 & 0 & 1 & 0.36 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0.045 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\theta_2 = {}^{\pi}/_{4}, \theta_4 = -{}^{\pi}/_{4}, \theta_6 = -{}^{\pi}/_{2}$$

$$T_{sb}(\theta) = Me^{[\mathcal{B}_2]\theta_2}e^{[\mathcal{B}_4]\theta_4}e^{[\mathcal{B}_6]\theta_6}$$

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

$$e^{-[\mathcal{B}_4]^{\pi}/_{4}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sin\frac{-\pi}{4}\begin{bmatrix} 0 & 0 & 1 & 0.36 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0.045 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 1 & 0.36 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0.045 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0.36 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0.045 \\ 0 & 0 & 0 & 0 \end{bmatrix}\left(1 - \cos\frac{-\pi}{4}\right) = \begin{bmatrix} 0.707 & 0 & -0.707 & -0.241 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & 0.707 & -0.137 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Barrett Technology’s WAM 7R robot arm

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [\mathcal{B}_6] = \begin{bmatrix} 0 & 0 & 1 & 0.06 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\theta_2 = {}^{\pi}/_{4}, \theta_4 = -{}^{\pi}/_{4}, \theta_6 = -{}^{\pi}/_{2}$$

$$T_{sb}(\theta) = Me^{[\mathcal{B}_2]\theta_2}e^{[\mathcal{B}_4]\theta_4}e^{[\mathcal{B}_6]\theta_6}$$

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$$

$$e^{-[\mathcal{B}_6]\pi/2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \sin\frac{-\pi}{2}\begin{bmatrix} 0 & 0 & 1 & 0.06 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 1 & 0.06 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0.06 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\left(1 - \cos\frac{-\pi}{2}\right) = \begin{bmatrix} 0 & 0 & -1 & -0.06 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -0.06 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Barrett Technology's WAM 7R robot arm

$$T_{sb}(\theta) = Me^{[\mathcal{B}_2]\pi/4}e^{-[\mathcal{B}_4]\pi/4}e^{-[\mathcal{B}_6]\pi/2}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.91 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & 0 & 0.707 & 0.6435 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & -0.267 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.707 & 0 & -0.707 & -0.241 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & 0.707 & -0.137 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & -0.06 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -0.06 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.707 & 0 & 0.707 & 0.6435 \\ 0 & 1 & 0 & 0 \\ -0.707 & 0 & 0.707 & 0.6435 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.707 & 0 & -0.707 & -0.241 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & -0.707 & -0.222 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 0.316 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.657 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary

- **Kinematics** is the study of motion without considering the force
- **Dynamics** study the motion with consideration of force
- **Forward kinematics** compute the manipulator **pose from joint** parameters (angles and translation)
- **Inverse kinematics** determine the desired joint parameters to achieve a desired pose
- Forward kinematics can be determined by (1) **geometry**, (2) **transformation matrix**, (3) **power of exponential** with screw axes
- **Power of exponent** (PoE) in **base form** determine the screws w.r.t. the base frame, and pre-multiply the matrix exponents
- Power of exponent (PoE) in **body form** determine the screws w.r.t. the body frame, and post-multiply the matrix exponents

Reading List

- Read Chapter 4 of Modern Robotics

To Do List

- Watch Chapter 4 videos of Modern Robotics on Coursera, or on YouTube https://www.youtube.com/playlist?list=PLggLP4f-rq02vX0OQQ5vrCxbJrzamYDfx